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POSTSCRIPT TO A TAUBERIAN THEOREM

By

S. MINAKSHISUNDARAM (*Waltair*) and

C. T. RAJAGOPAL (*Tambaram*)

[Received 18 December 1946]

In a previous note we stated two theorems with two-sided Tauberian O-conditions.* The one-sided analogue of one of these theorems having been given in another note,† we record, for the sake of completeness, the analogue of the other, using the notation of our earlier notes:

$\sum_{n=0}^{\infty} a_n$ is a real series; $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$; $\lambda_n \to \infty$;

$$A^r(\omega) = r \int_0^{\omega} (\omega - t)^{r-1} A(t)\, dt \quad (r > 0),$$

$$A^0(t) = A(t) = \sum_{\nu=0}^{n} a_\nu \quad (\lambda_n \leqslant t < \lambda_{n+1}).$$

THEOREM. *Suppose that*

(i) $\phi(x)$, $\psi(x)$ *are positive functions of* $x > 0$ *such that*

$$\theta(x) = \{\phi(x)/\psi(x)x^{1/k}\}^{1/(r+1/k')} \leqslant x, \tag{1}$$

where $k > 1$, $\dfrac{1}{k} + \dfrac{1}{k'} = 1$, $r > 0$,

$$\vartheta(x) = \phi(x)/\{\theta(x)\}^r;$$

(ii) $\phi(x)$ *is non-decreasing; there are constants* $H \geqslant 1$, $0 < \eta < 1$, *such that*

$$\frac{\phi(x')}{\phi(x)} \leqslant H \quad \textit{when} \quad 0 < x' - x \leqslant \eta\theta(x), \tag{2}$$

$$\frac{\psi(x')}{\psi(x)} \leqslant H \quad \textit{when} \quad 0 < |x' - x| \leqslant \eta\theta(x); \tag{3}$$

(iii) *either* $\lambda_n - \lambda_{n-1} \leqslant \eta\theta(\lambda_n)$ *or* $a_n = O_R[\vartheta(\lambda_n)]$ $(n \to \infty)$; (4)

(iv) $$\sum_{\nu=1}^{n} (|a_\nu| + a_\nu)^k \lambda_\nu^k (\lambda_\nu - \lambda_{\nu-1})^{1-k} \leqslant K\{\psi(\lambda_n)\}^k \lambda_n^{k+1}; \tag{5}$$

* *Quart. J. of Math.* (Oxford), 17 (1946), 153–61, Theorems A, A′. This note will be referred to as T_1.

† 'An extension of a Tauberian theorem of L. J. Mordell', *Proc. London Math. Soc.* (unpublished). This will be referred to as T_2.

(v) $$A^r(\omega) = O[\phi(\omega)] \quad (\omega \to \infty). \tag{6}$$

Then $$A(\lambda_n) = O[\vartheta(\lambda_n)] \quad (n \to \infty). \tag{7}$$

Proof. Since we can prove that $A(\lambda_n) = O_L[\vartheta(\lambda_n)]$,* it suffices to establish the remaining half of the conclusion: $A(\lambda_n) = O_R[\vartheta(\lambda_n)]$. The reader can do this for himself with the help of the lemma of Hardy and Riesz already used to prove the first half of the conclusion† and the following lemmas.

LEMMA I. *For any function $F(x)$ and $\phi > 0$, let*

$$\Delta^n_{-\phi} F(x) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} F(x-\nu\phi) \quad (n = 1, 2, \ldots).$$

Then $$\int_{x-\phi}^{x} F(x-t)\Delta^\nu_{-\phi} G(t)\, dt = \Delta^\nu_{-\phi} \int_{x-\phi}^{x} F(x-t)G(t)\, dt,$$

$F(t)$, $G(t)$ being supposed integrable.

LEMMA II.‡

$$\Delta^p_{-\phi} A^p(t_0) = \Gamma(p+1) \int_{t_0-\phi}^{t_0} dt_1 \int_{t_1-\phi}^{t_1} dt_2 \int_{t_2-\phi}^{t_2} \ldots \int_{t_{p-1}-\phi}^{t_{p-1}} A(t)\, dt \quad (p = 1, 2, \ldots).$$

LEMMA III. *Under the conditions* (1), (2), (3), (4), (5),

$$A(\lambda_m) - A(t) < C\vartheta(\lambda_m),$$

where C is a constant (independent of m) and

$$\lambda_m - \epsilon\theta(\lambda_m) \leqslant t < \lambda_m \quad (0 < \epsilon \leqslant \eta).$$

Lemmas I and II being readily proved by induction, we supply only the

Proof of Lemma III. Suppose that

$$\lambda_q \leqslant \lambda_m - \epsilon\theta(\lambda_m) < \lambda_{q+1} \quad (q < m), \tag{8}$$
$$\lambda_\nu \leqslant t < \lambda_{\nu+1} \quad (q+1 \leqslant \nu+1 \leqslant m).$$

Then we have

$$A(\lambda_m) - A(t) = a_{\nu+1} + \ldots + a_m \leqslant |a_{\nu+1}| + a_{\nu+1} + \ldots + |a_m| + a_m$$
$$= \sum_{n=1}^{m-\nu} (|a_{\nu+n}| + a_{\nu+n})\lambda_{\nu+n}(\lambda_{\nu+n} - \lambda_{\nu+n-1})^{(1-k)/k}(\lambda_{\nu+n} - \lambda_{\nu+n-1})^{1/k'}/\lambda_{\nu+n}$$
$$\leqslant \left\{\sum_{n=1}^{m-\nu} (|a_{\nu+n}| + a_{\nu+n})^k \lambda^k_{\nu+n}(\lambda_{\nu+n} - \lambda_{\nu+n-1})^{1-k}\right\}^{1/k} \left\{\sum_{n=1}^{m-\nu} \frac{\lambda_{\nu+n} - \lambda_{\nu+n-1}}{\lambda^{k'}_{\nu+n}}\right\}^{1/k'}$$

* As in T_1, §§ 3, 2.2, 2.4 (i)

† T_1, Lemma 1.

‡ L. S. Bosanquet, *J. of London Math. Soc.* 18 (1943), 239–48.

by Hölder's inequality; and so we obtain from (5)

$$A(\lambda_m)-A(t) \leqslant K^{1/k}\psi(\lambda_m)\lambda_m^{1+1/k}\frac{(\lambda_m-\lambda_\nu)^{1/k'}}{\lambda_{\nu+1}}. \tag{9}$$

Also, if $\nu \geqslant q+1$, (8) and (1) give

$$\lambda_m-\lambda_\nu \leqslant \lambda_m-\lambda_{q+1} < \epsilon\theta(\lambda_m),$$

$$\frac{\psi(\lambda_m)\lambda_m^{1+1/k}}{\lambda_{\nu+1}} \leqslant \frac{\psi(\lambda_m)\lambda_m^{1+1/k}}{\lambda_{q+2}} < \frac{\psi(\lambda_m)\lambda_m^{1/k}}{1-\epsilon},$$

which in conjunction with (9) lead to

$$A(\lambda_m)-A(t) < K^{1/k}\frac{\epsilon^{1/k'}}{1-\epsilon}\psi(\lambda_m)\lambda_m^{1/k}\{\theta(\lambda_m)\}^{1/k'}$$

$$= K^{1/k}\frac{\epsilon^{1/k'}}{1-\epsilon}\vartheta(\lambda_m) \leqslant K^{1/k}H\frac{\epsilon^{1/k'}}{1-\epsilon}\vartheta(\lambda_m). \tag{10}$$

Next, the argument which establishes (9) shows that

$$a_{q+1} \leqslant K^{1/k}\psi(\lambda_{q+1})\lambda_{q+1}^{1+1/k}\frac{(\lambda_{q+1}-\lambda_q)^{1/k'}}{\lambda_{q+1}}.$$

If the first alternative of (4) is given, we get from the above

$$a_{q+1} \leqslant K^{1/k}\psi(\lambda_{q+1})\lambda_{q+1}^{1/k}\eta^{1/k'}\{\theta(\lambda_{q+1})\}^{1/k'} = K^{1/k}\eta^{1/k'}\vartheta(\lambda_{q+1}), \tag{11}$$

where, in virtue of (8) and the hypothesis (ii),

$$\vartheta(\lambda_{q+1}) = \{\phi(\lambda_{q+1})\}^{\frac{1/k'}{r+1/k'}}\{\psi(\lambda_{q+1})\}^{\frac{r}{r+1/k'}}\lambda_{q+1}^{\frac{r/k}{r+1/k'}}$$

$$\leqslant \{\phi(\lambda_m)\}^{\frac{1/k'}{r+1/k'}}H^{\frac{r}{r+1/k'}}\{\psi(\lambda_m)\}^{\frac{r}{r+1/k'}}\lambda_m^{\frac{r/k}{r+1/k'}},$$

and consequently $$a_{q+1} \leqslant K^{1/k}\eta^{1/k'}H\vartheta(\lambda_m). \tag{12}*$$

Hence with (10), when $\nu = q$,

$$A(\lambda_m)-A(t) = a_{q+1}+(a_{q+2}+\ldots+a_m)$$

$$< K^{1/k}H\left(\eta^{1/k'}+\frac{\epsilon^{1/k'}}{1-\epsilon}\right)\vartheta(\lambda_m). \tag{13}$$

Thus, with the first alternative of (4), our lemma follows from (10) in the case $\nu \geqslant q+1$ and (13) in the case $\nu = q$.

The lemma remains valid, with the second alternative of (4), since (11) holds even then, although with a constant different from $K^{1/k}\eta^{1/k'}H$.

* In obtaining (12) we have assumed that $\phi(x)$ is non-decreasing. Alternatively we may assume that $\phi(x')/\phi(x) \leqslant H$ when $-\eta\theta(x) \leqslant x'-x < 0$.

Remarks. (i) There is a version of the theorem with (4), (6), (7) replaced respectively by:*

$$\textit{either} \quad \lambda_n - \lambda_{n-1} = o[\theta(\lambda_n)] \quad \textit{or} \quad a_n = o_R[\vartheta(\lambda_n)] \quad (n \to \infty), \tag{4'}$$

$$A^r(\omega) = o[\phi(\omega)] \quad (\omega \to \infty). \tag{6'}$$

$$A(\lambda_n) = o[\vartheta(\lambda_n)] \quad (n \to \infty). \tag{7'}$$

(ii) The particular case $\phi(x) = x^\beta$, $\psi(x) = x^\alpha$ of the preceding result leads to a one-sided Tauberian theorem corresponding to a two-sided theorem of Ganapathy Iyer.† The further particular case $r = \beta = -\alpha = 1$ is a theorem of Szász.‡

(iii) If, in our theorem, r is an integer, the supposition that $\phi(x)$ is non-decreasing may be dispensed with, provided that (2) is replaced by $\phi(x')/\phi(x) \leqslant H$ when $0 < |x'-x| \leqslant \eta\theta(x)$.§

If, further, $\lambda_n = n$, the first alternatives of (4') and (4) necessarily hold since we may work with the supposition $\theta(x) = O(x)$ instead of $\theta(x) \leqslant x$. We have thus Cesàro-summability results—with (4) and (4') left out—following from our theorem and its o-version in (i) above. These results are formal extensions of certain theorems of Mordell and Kloosterman.‖

* The replacement of (6) by (6') is justified as in T_2, Corollary 1.1, the other replacements requiring only obvious modifications.

† *Annals of Math.* 36 (1935), 100–16, Theorem 4; and *J. of Indian Math. Soc.* (New Series), 2 (1936), 147–55, Theorem 2.

‡ e.g. *Trans. American Math. Soc.* 39 (1936), 117–30, Theorem 4.

§ See T_2, § 2.4 (ii).

‖ *J. of London Math. Soc.* 3 (1928), 86–9, and 15 (1940), 91–6.

ON THE NUMBER OF ABELIAN GROUPS OF A GIVEN ORDER

By D. G. KENDALL (*Oxford*) and
R. A. RANKIN (*Cambridge*)

[Received 26 January 1947]

1. LET $a(n)$ be the number of essentially distinct Abelian groups of order n. From a well-known result on the generation of Abelian groups,† it follows that $a(n)$ is equal to the number of ways in which n can be expressed as the product of powers of primes (the order of the factors being irrelevant). Thus there are 14 Abelian groups of order 800, a typical factorization being $800 = 5^2.2^2.2.2.2$.

The function $a(n)$ is clearly multiplicative in the restricted sense, i.e.

$$a(mn) = a(m)a(n) \quad \text{for} \quad (m,n) = 1,$$

and it is therefore uniquely determined when $a(p^\lambda)$ is known, where p is any prime and λ is any non-negative integer. Now $a(p^\lambda)$ is equal to the number of partitions of λ into positive parts, and so

$$\sum_{\lambda=0}^{\infty} a(p^\lambda)p^{-\lambda s} = \prod_{\nu=1}^{\infty} (1-p^{-\nu s})^{-1} \tag{1}$$

when $\sigma = \Re s > 0$. It follows that

$$F(s) \equiv \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{\nu=1}^{\infty} \zeta(\nu s), \tag{2}$$

where $\zeta(s)$ is Riemann's zeta function, the infinite series and product being absolutely convergent for $\sigma > 1$. The function $F(s)$ is therefore a regular function of s in the half-plane $\sigma > 1$, and, from its definition as an infinite product, it is easily seen that it is in fact regular in the half-plane $\sigma > 0$ with the exception of simple poles at the points $s = 1/r$ $(r = 1, 2, \ldots)$.

Write $$A(x) = \sum_{n \leqslant x} a(n).$$

We prove the following

THEOREM. *For large* x,

$$A(x) = \alpha x - \beta x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log x), \tag{3}$$

† See, for example, A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung* (Berlin, 1937), 49–51.

where $\alpha = \zeta(2)\zeta(3)\zeta(4)\ldots = 2{\cdot}29485\ldots,$

and $\beta = -\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})\zeta(\tfrac{4}{2})\zeta(\tfrac{5}{2})\ldots = 14{\cdot}6\ldots.$

Thus, on the average, there are about 2 to 3 distinct Abelian groups of any given order.

2. In order to prove the theorem we need three lemmas. Write

$$F_1(s) = \zeta(s)\zeta(2s) = \sum_{n=1}^{\infty} b_n n^{-s}, \tag{4}$$

$$F_2(s) = \prod_{\nu=3}^{\infty} \zeta(\nu s) = \zeta(3s)F_3(s) = \sum_{n=1}^{\infty} c_n n^{-s}, \tag{5}$$

and $$F_3(s) = \sum_{n=1}^{\infty} d_n n^{-s}.$$

Also put $$B(x) = \sum_{n\leqslant x} b_n, \qquad C(x) = \sum_{n\leqslant x} c_n.$$

LEMMA 1. *For large* x,

$$B(x) = x\zeta(2)+x^{\frac{1}{2}}\zeta(\tfrac{1}{2})+O(x^{\frac{1}{3}}).$$

We obtain this result by applying a famous theorem of Landau† to the function $F_1(s)$. The equation (4) defines $F_1(s)$ as a regular function of s over the whole plane, with the exception of the points $s = 1$ and $s = \frac{1}{2}$, where it has simple poles with residues $\zeta(2)$ and $\frac{1}{2}\zeta(\frac{1}{2})$ respectively. Also, from the functional equation for $\zeta(s)$, we have

$$\pi^{-\frac{3}{2}s}\Gamma(\tfrac{1}{2}s)\Gamma(s)F_1(s) = \pi^{-1+\frac{3}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\Gamma(1-s)\zeta(1-s)\zeta(1-2s).$$

Accordingly, $F_1(s)$ satisfies the conditions I to VII, VIII′, and IX of Landau's theorem with

$$\beta = 1, \qquad g = 0, \qquad \eta = 1, \qquad \kappa = \tfrac{1}{3},$$

and this gives the required result.

LEMMA 2. *For large* x,

$$C(x) = O(x^{\frac{1}{3}}).$$

Considerably more than this can be shown, but this is all that we require. It is clear, from (5), that the Dirichlet series for $F_3(s)$ is absolutely convergent for $\sigma > \frac{1}{4}$. Hence

$$C(x) = \sum_{m^3n\leqslant x} d_n \leqslant \sum_{n\leqslant x} \left(\frac{x}{n}\right)^{\frac{1}{3}} d_n \leqslant x^{\frac{1}{3}} \sum_{n=1}^{\infty} d_n n^{-\frac{1}{3}} = O(x^{\frac{1}{3}}).$$

† E. Landau, 'Über die Anzahl der Gitterpunkte in gewissen Bereichen', *Göttinger Nachr.* (1915), 209–43.

LEMMA 3. *We have*

$$\sum_{n\leqslant x} c_n n^{-\alpha} = F_2(\alpha)+O(x^{\frac{1}{3}-\alpha}) \quad (\alpha > \tfrac{1}{3}),$$

and

$$\sum_{n\leqslant x} c_n n^{-\frac{1}{3}} = O(\log x).$$

The first part of the lemma follows at once from Lemma 2 by writing

$$\sum_{n\leqslant x} c_n n^{-\alpha} = F_2(\alpha) - \sum_{n>x} c_n n^{-\alpha},$$

since the Dirichlet series for $F_2(s)$ is absolutely convergent for $\sigma > \frac{1}{3}$. The second part is a direct consequence of Lemma 2.

3. Proof of the theorem

We have, from (2), (4), (5), and Lemma 1,

$$A(x) = \sum_{mn\leqslant x} b_m c_n = \sum_{n\leqslant x} B\left(\frac{x}{n}\right)c_n$$

$$= \sum_{n\leqslant x}\left\{\frac{x}{n}\zeta(2)+\left(\frac{x}{n}\right)^{\frac{1}{2}}\zeta(\tfrac{1}{2})\right\}c_n + O\left\{\sum_{n\leqslant x}\left(\frac{x}{n}\right)^{\frac{1}{3}}c_n\right\}.$$

It follows, from Lemma 3 with $\alpha = 1, \frac{1}{2}$, that

$$A(x) = x\zeta(2)\{F_2(1)+O(x^{-\frac{2}{3}})\}+x^{\frac{1}{2}}\zeta(\tfrac{1}{2})\{F_2(\tfrac{1}{2})+O(x^{-\frac{1}{6}})\}+O(x^{\frac{1}{3}}\log x)$$

$$= x\zeta(2)F_2(1)+x^{\frac{1}{2}}\zeta(\tfrac{1}{2})F_2(\tfrac{1}{2})+O(x^{\frac{1}{3}}\log x),$$

which, by (5), is the required result.

It may be noted that (3) does not agree with a result obtained by Wintner.† Starting from the wrong generating function, namely, $\zeta(s)\zeta(2s)\zeta(3s)/\zeta(6s)$ in place of $F(s)$, Wintner has obtained for the 'mean value' of $a(n)$

$$\zeta(2)\zeta(3)/\zeta(6) = 1{\cdot}94359\ldots.$$

This is lower than the correct value 2·29... given by (3), and in fact Wintner's calculation gives the 'mean value' of the arithmetical function

$$a^*(n) = \alpha\beta\ldots\lambda \quad (n > 1), \qquad a^*(1) = 1,$$

instead of

$$a(n) = P(\alpha)P(\beta)\ldots P(\lambda) \quad (n > 1), \qquad a(1) = 1,$$

where $n = p_1^{\alpha}p_2^{\beta}\ldots p_l^{\lambda}$ is the unique expression for n as the product of powers of different primes, and $P(m)$ is the number of partitions of m into positive parts.

† A. Wintner, *Eratosthenian Averages* (Baltimore, 1943), 46–7.

4. The distribution of values of $a(n)$

First let $a(n)$ be any non-negative integer-valued function of the positive integer n; then we define the *asymptotic frequency distribution* of the values of $a(n)$ as the sequence of numbers

$$P_m = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a(n)=m \\ n\leqslant N}} 1 \quad (m = 0, 1, 2, \ldots), \tag{6}$$

if these limits exist. Thus P_m is the limit, as N becomes large, of the frequency with which $a(n)$ assumes the value m when n ranges from 1 to N. The limits P_m may not exist, or they may exist and yet may not constitute a true frequency distribution; i.e. we may have

$$0 \leqslant \sum_{m=0}^{\infty} P_m < 1.$$

For example, if $a(n) \equiv n$, then $P_m = 0$ for all m. Again, we may have

$$\sum_{m=0}^{\infty} P_m = 1,$$

and yet the *mean* of the asymptotic distribution,

$$E(m) \equiv \sum_{m=0}^{\infty} mP_m,$$

may differ from the limit of the mean of the finite distribution,

$$\lim_{N\to\infty} E_N(m) \equiv \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a(n)=m \\ n\leqslant N}} m = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a(n). \tag{7}$$

Thus, if $a(n) = \sqrt{n}$ when n is a perfect square, and $a(n) = 0$ for all other values of n, we shall have

$$P_0 = 1, \qquad P_m = 0 \quad (m > 0),$$

so that

$$\sum_{m=0}^{\infty} P_m = 1 \quad \text{and} \quad \sum_{m=0}^{\infty} mP_m = 0,$$

while

$$E_N(m) = \frac{[\sqrt{N}]\{[\sqrt{N}]+1\}}{2N},$$

which has the limit $\frac{1}{2}$ as N tends to infinity.

These difficulties must always be borne in mind when one is working with the asymptotic frequency distribution of the values of an arithmetical function; they arise from the fact that one is then concerned with the *limit* of a distribution, which need not be a true distribution itself, nor (if it is a true distribution) need its moments

be the limits of the moments of the original distribution. This situation is clarified by the so-called 'second limit-theorem' of the theory of probability, in the form given by M. Fréchet and J. Shohat,† and A. Wintner.‡

It may be noted, however, that one will always have

$$\sum_{m=0}^{\infty} P_m \leqslant 1 \quad \text{and} \quad \sum_{m=0}^{\infty} mP_m \leqslant \lim_{N\to\infty} E_N(m),$$

whenever the P_m and the limit on the right-hand side exist, the proofs of these facts being elementary.

If now we take $a(n)$ to be the number of Abelian groups of order n, it will be seen that the theorem of § 1 includes the statement

$$\lim_{x\to\infty} A(x)/x = \lim_{N\to\infty} E_N(m) = \alpha = 2{\cdot}29485\ldots.$$

The function $a(n)$ is of special interest in being one of the few naturally occurring arithmetical functions with finite 'mean values' for which the asymptotic frequency distribution $\{P_m\}$ can be calculated explicitly. The only other example known to us is the function $r(n)$ which enumerates the representations of the integer n as the sum of two squares. It has been shown by Landau§ that the number of integers n (not exceeding x) for which $r(n) > 0$ is asymptotically

$$K\frac{x}{\sqrt{(\log x)}},$$

where K is a known constant, and so $P_0 = 1$, while $P_m = 0$ $(m \geqslant 1)$. For the function $r(n)$, therefore,

$$\sum_{m=0}^{\infty} P_m = 1,$$

but it will be noticed that

$$\sum_{m=0}^{\infty} mP_m = 0 \neq \lim_{N\to\infty} E_N\{r(n)\},$$

for the limit on the extreme right is well known to have the value π.

5. A necessary lemma

To assist in the calculation of the P_m for the function $a(n)$ we need the following well-known lemma. It is included within a theorem

† 'A Proof of the Generalized Second Limit-Theorems in the Theory of Probability', *Trans. American Math. Soc.* 33 (1931), 533–43.

‡ 'Über den Konvergenzsatz der Mathematischen Statistik', *Math. Zeits.* 28 (1928), 476–80.

§ See, for example, G. H. Hardy, *Ramanujan* (Cambridge, 1940), 60–3.

of A. Wintner,† but his result is unnecessarily deep for our present purpose, and it is therefore convenient to have a simple and direct proof of the result needed here.

LEMMA 4. *Let*

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \zeta(s)\sum_{n=1}^{\infty} g(n)n^{-s} \quad (\textit{all } \sigma > 1),$$

and let the Dirichlet series on the right be absolutely convergent when $\sigma = 1$. *Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n) = \sum_{n=1}^{\infty}\frac{g(n)}{n}.$$

An application of the lemma at once gives the mean-value result contained in the theorem of § 1. We set

$$\sum_{n=1}^{\infty} g(n)n^{-s} = \zeta(2s)\zeta(3s)\zeta(4s)\ldots;$$

this Dirichlet series evidently converges absolutely for $s = 1$, and so, from (2),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} a(n) = \zeta(2)\zeta(3)\zeta(4)\ldots,$$

where $a(n)$ is the number of Abelian groups of order n.

The lemma is easily proved. We have $f(n) = \sum_{m|n} g(m)$, and so

$$\sum_{n=1}^{N} f(n) = \sum_{m=1}^{N} g(m)\left[\frac{N}{m}\right],$$

and

$$\frac{1}{N}\sum_{n=1}^{N} f(n) = \sum_{m=1}^{N}\frac{g(m)}{m} + O\left\{\frac{1}{N}\sum_{m=1}^{N}|g(m)|\right\}.$$

Now write

$$G(m) = \sum_{r=m}^{\infty}\frac{|g(r)|}{r},$$

so that $G(m)$ tends to zero as m tends to infinity. Then

$$\frac{1}{N}\sum_{m=1}^{N}|g(m)| = \frac{1}{N}\sum_{m=2}^{N} G(m) - G(N+1) + G(1)/N,$$

which tends to zero as N tends to infinity, in virtue of the consistency theorem‡ for $(C, 1)$ summability.

† See the result (XIII) on page 25 of Wintner's monograph mentioned in § 3. Another result more powerful than and including Lemma 4 can be deduced from Satz 1 of Landau's paper "Über einige neuere Grenzwertsätze", *Rend. Circ. Mat. Palermo*, 34 (1912), 121–31.

‡ See, for example, E. C. Titchmarsh, *The Theory of Functions* (Oxford, 1932), 10.

This completes the proof; it should be noted that there is no need for the functions $f(n)$, $g(n)$ to be multiplicative.

6. Calculation of the P_m

An integer n can be expressed uniquely in the form

$$n = p_1^{\alpha} p_2^{\beta} \ldots p_l^{\lambda} Q,$$

where Q is square-free (i.e. has no square divisor) and is prime to each of the (different) primes $p_1, p_2, \ldots, p_l$, while $\alpha, \beta, \ldots, \lambda$ are all positive integers *greater than unity*. The integer n will then be said to have the *constitution* $(\alpha, \beta, \ldots, \lambda)$, and of course we shall have

$$a(n) = P(\alpha)P(\beta)\ldots P(\lambda).$$

From the following short table of the partition function $P(n)$,

$n =$	1	2	3	4	5	6	7
$P(n) =$	1	2	3	5	7	11	15,

it will be seen that each of the values $m = 1, 2, \ldots, 12, 14$ of the function $a(n)$ arises from values of n having a unique constitution. There are no values of n which make $a(n) = 13$, while $a(n) = 15$ if n has either of the two constitutions $p_1^3 p_2^4 Q$ or $p_1^7 Q$. Thus, in the asymptotic frequency distribution for $a(n)$, $P_{13} = 0$; we shall calculate the values of P_m for $1 \leqslant m \leqslant 15$.

Let $f(n) = 1$ if n has the constitution $(\alpha, \beta, \ldots, \lambda)$, and let $f(n) = 0$ for all other values of n. Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{(1)} p_1^{-\alpha s} p_2^{-\beta s} \ldots p_l^{-\lambda s} \prod{}' (1+p^{-s}) \quad (\sigma > 1), \tag{8}$$

where in the infinite product the factors involving the primes $p_1, p_2, \ldots, p_l$ are all omitted, and where the summation is extended over all sets of different primes $p_1, p_2, \ldots, p_l$ which give distinct values to the product

$$p_1^{\alpha} p_2^{\beta} \ldots p_l^{\lambda}.$$

Thus, for example, if $\alpha = \beta$, the terms involving $p_1 = a$, $p_2 = b$ and $p_1 = b$, $p_2 = a$ will not *both* be counted. The expression on the right-hand side of (8) is absolutely convergent when $\sigma > 1$, because each of $\alpha, \beta, \ldots, \lambda$ is not less than 2, and it can be written in the form

$$\frac{\zeta(s)}{\zeta(2s)} \sum_{(1)} \prod_{r=1, \rho=\alpha}^{r=l, \rho=\lambda} \frac{1}{p_r^{\rho s}(1+p_r^{-s})},$$

the same conventions as before being understood in regard to the sign of summation. The lemma will apply if we take

$$\sum_{m=1}^{\infty} \frac{g(m)}{m^s} = \frac{1}{\zeta(2s)} \sum_{(1)} \prod_{r=1,\rho=\alpha}^{r=l,\rho=\lambda} \frac{1}{p_r^{\rho s}(1+p_r^{-s})},$$

this Dirichlet series being absolutely convergent for $\sigma = 1$ because each value of ρ is greater than unity. One thus obtains the following result: *for the values m assumed by $a(n)$, the asymptotic frequency distribution is given by*

$$P_0 = 0, \qquad P_1 = 6/\pi^2,$$

and

$$P_m = \frac{6}{\pi^2} \sum_{(2)} \sum_{(1)} \prod_{r=1,\rho=\alpha}^{r=l,\rho=\lambda} \frac{1}{p_r^{\rho}(1+p_r^{-1})} \quad (m > 1), \tag{9}$$

where $\sum_{(1)}$ has the meaning already explained, and $\sum_{(2)}$ is to be extended over all constitutions $(\alpha, \beta, \ldots, \lambda)$ containing any finite number of terms, l, such that

$$P(\alpha)P(\beta)\ldots P(\lambda) = m,$$

$P(\alpha)$ being the number of partitions of α into positive parts.

The value of P_1 is obtained by applying the lemma to

$$\sum_{(Q)} \frac{1}{n^s} = \prod \left(1+\frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}} \quad (\sigma > 1),$$

where the first summation is over all square-free numbers, and where $\mu(n)$ is the Möbius function. This is equivalent to the familiar calculation of the 'probability' that a number shall be square-free.

It can now be verified that with the above values for P_m,

$$\sum_{m=0}^{\infty} P_m = 1,$$

and

$$\sum_{m=0}^{\infty} mP_m = \zeta(2)\zeta(3)\zeta(4)\ldots = \lim_{N\to\infty} E_N(m).$$

It will be recalled that the numbers $\{P_m\}$ associated with the function $r(n)$ were shown (at the end of § 4) to possess only the first of these two properties.

7. Numerical values for the P_m

The calculation of numerical values for the P_m, when $m \leqslant 15$, is quite straightforward, if use is made of the infinite series

$$\Sigma_\alpha \equiv \sum p^{-\alpha}, \qquad U_\alpha \equiv \sum p^{-\alpha}(1+1/p)^{-1} = \Sigma_\alpha - \Sigma_{\alpha+1} + \Sigma_{\alpha+2} - \ldots,$$

$$V_\alpha \equiv \sum p^{-\alpha}(1+1/p)^{-2} = U_\alpha - U_{\alpha+1} + U_{\alpha+2} - \ldots,$$

and

$$W_\alpha \equiv \sum p^{-\alpha}(1+1/p)^{-3} = V_\alpha - V_{\alpha+1} + V_{\alpha+2} - \ldots.$$

The series Σ_α were evaluated for even values of α by Euler in 1748; for our purposes the most convenient table is that of C. W. Merrifield.† This lists the values of Σ_α for all values of α from 2 to 35, each being given to 15 places of decimals. The value of Σ_3 was corrected by J. W. L. Glaisher,‡ who pointed out that it should commence with the figures 0·17476.... No other errors are mentioned in the 'Index' of A. Fletcher, J. C. P. Miller, and L. Rosenhead.§

The following table gives, for $1 \leqslant a(n) = m \leqslant 15$, the constitution of n, the value of P_m, and the actual number of values of n for which

$$a(n) = m \quad \text{and} \quad n \leqslant 1000.$$

The entries in the last column are based on Volume V (1935) of the British Association 'Mathematical Tables'. No attempt was made to compute the values of P_m with all the precision available, as we merely wished to obtain a general idea of the form of the distribution for the first few values of m.

$m = a(n)$	*Constitution of* n	P_m	*Number of* $n \leqslant 1000$ *for which* $a(n) = m$
1	Q	0·6079	608
2	$p_1^2 Q$	2008	204
3	$p_1^3 Q$	0742	76
4	$p_1^2 p_2^2 Q$	0221	20
5	$p_1^4 Q$	0321	33
6	$p_1^2 p_2^3 Q$	0145	14
7	$p_1^5 Q$	0147	15
8	$p_1^2 p_2^2 p_3^2 Q$	0011	1
9	$p_1^3 p_2^3 Q$	0022	2
10	$p_1^2 p_2^4 Q$	0059	5
11	$p_1^6 Q$	0070	8
12	$p_1^2 p_2^2 p_3^3 Q$	0009	0
13	nil	nil	nil
14	$p_1^2 p_2^5 Q$	0026	3
15	$p_1^3 p_2^4 Q$	0016	2
	$p_1^7 Q$	0034	4

There are five integers less than or equal to 1000 which belong to none of the above classes; they are 256 ($= 2^8$), 512 ($= 2^9$), 576 ($= 2^6 . 3^2$), 768 ($= 2^8 . 3$), and 864 ($= 2^5 . 3^3$).

† 'The Sum of the Series of the Reciprocals of the Prime Numbers and their Powers', *Proc. Royal Soc.* 33 (1881), 10.

‡ 'On the Series $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$', *Quart. J. of Math.* 25 (1891), 373.

§ *An Index of Mathematical Tables* (London, 1946).

A comparison of the third and fourth columns is of interest in itself, and it also provides a useful check on the calculations. Since

$$\sum_{m=0}^{15} P_m = 0{\cdot}9910$$

(with an error of at most one unit in the last place), more than 99 per cent. of the distribution is covered by the table. On the other hand, since

$$\sum_{m=0}^{15} mP_m = 1{\cdot}958, \quad \text{and} \quad \sum_{m=0}^{\infty} mP_m = 2{\cdot}295,$$

the rate of fall-off of P_m, as m tends to infinity, is evidently not very rapid.

From the data thus collected it is possible to calculate the exact value of $A(1000)$, and to compare it with that given by the formula (3). We find, when $x = 1000$,

$$A(x) = 2091,$$

and

$$\alpha x - \beta x^{\frac{1}{2}} = 2295 - 462 = 1833 \quad \text{(last digit uncertain).}$$

The error term $O(x^{\frac{1}{2}} \log x)$ is $O(69)$, so that even though the suppressed constant is unknown, it is not really surprising that the approximation should be so poor when $x = 1000$.

8. The order of $a(n)$ for large n

The preceding results concerning the asymptotic frequency distribution of the values of $a(n)$ naturally conceal any dependence of $a(n)$ upon n. It seems therefore fitting to conclude with some remarks about the order of $a(n)$ for large n.

Since $a(n) = P(\alpha)$ when n is a prime power p^α, it is clear that $a(n)$ is unbounded. On the other hand, it can be shown that

$$a(n) = O(n^\delta)$$

for every fixed $\delta > 0$. In fact, there exists a positive constant c such that

$$a(n) = O\{\exp(c \log n/\log\log n)\} \tag{10}$$

for all sufficiently large n.

If n is square-free, (10) is trivial since $a(n) = 1$. Suppose therefore that $n = Qn_1$, where Q is the greatest square-free factor of n such that $(Q, n_1) = 1$, and that $n_1 > 1$. Then n_1 can be written in the form $q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_m^{\alpha_m}$, where $q_1, q_2, \ldots, q_m$ are primes, $q_1 < q_2 < \ldots < q_m$, $m \geqslant 1$ and $\alpha_i \geqslant 2$ $(i = 1, 2, \ldots, m)$. Write

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m},$$

where $p_1, p_2, \ldots, p_m$ are the first m primes in ascending order of magnitude. Then, clearly, $n \geqslant n_1 \geqslant N \geqslant 4$. Since

$$a(n) = a(n_1) = a(N)$$

and $\log x/\log\log x$ is an increasing function of x for $x > e^e$, it suffices to show that

$$a(N) \leqslant \exp(c \log N/\log\log N). \tag{11}$$

Now it is easily deduced from Chebyshev's inequalities,

$$A_1 \frac{x}{\log x} \leqslant \pi(x) \leqslant A_2 \frac{x}{\log x} \quad (x \geqslant 2),$$

that positive numbers a_1, a_2 exist such that

$$\log N \geqslant 2 \sum_{i=1}^{m} \log p_i \geqslant a_1 p_m,$$

and

$$\sum_{i=1}^{m} \frac{1}{\log p_i} \leqslant \frac{a_2 p_m}{(\log p_m)^2},$$

for all $m \geqslant 1$. From these two results it follows that a positive constant b exists such that

$$\sum_{i=1}^{m} \frac{1}{\log p_i} \leqslant \frac{b^2 \log N}{(\log\log N)^2} \quad (N \geqslant 4).$$

Again, it can be shown by elementary methods† that

$$P(\alpha) \leqslant e^{B\sqrt{\alpha}} \quad (\alpha \geqslant 1),$$

where B is an absolute constant. Hence we have, by Schwarz's inequality,

$$a(N) \leqslant \exp\Big(B \sum_{i=1}^{m} \alpha_i^{\frac{1}{2}}\Big) \leqslant \exp\Big\{B\Big(\sum_{i=1}^{m} \alpha_i \log p_i\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{m} 1/\log p_i\Big)^{\frac{1}{2}}\Big\}$$

$$\leqslant \exp(Bb \log N/\log\log N),$$

which proves (11) with $c = Bb$. By the use of the prime-number theorem and the asymptotic formula for $P(n)$ it is, no doubt, possible to obtain a more precise upper bound for $a(n)$ than that given by (10), but it is worth remarking that the exponent $c \log n/\log\log n$ cannot be replaced by a number of lower order. For example, if $n = p_1^2 p_2^2 \ldots p_m^2$, we can show that

$$a(n) = 2^m = \exp\{\tfrac{1}{2} \log 2 \log n(1+o(1))/\log\log n\}$$

for large n.

† See, for example, G. H. Hardy, loc. cit., § 8.3.

Such values of n are, however, very exceptional, and in fact for all $\epsilon > 0$,

$$a(n) < (\log n)^{C+\epsilon} \tag{12}$$

for 'almost all' n, where C is the constant $2\pi/\sqrt{6} = 2{\cdot}565\ldots$. More precisely, we shall show that the number of integers n (not exceeding x) for which

$$\log a(n) > C \log\log n + (\log\log n)^{\frac{1}{2}+\delta},$$

is $o(x)$ as x tends to infinity, for every $\delta > 0$.

It can be proved† by a slight refinement of Hardy's methods that

$$\log P(\alpha) < C\sqrt{\alpha} \tag{13}$$

for all $\alpha \geqslant 1$, where the constant C (having the value quoted above) is the 'best possible'. Thus

$$\log a(n) = \sum \log P(\alpha_i) < C \sum \sqrt{\alpha_i},$$

where now $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, $\alpha_i \geqslant 1$ for every i, and the summations are to be taken over all the k different prime factors of n. Since, in the notation of Hardy and Wright,‡

$$\omega(n) \equiv k = \sum 1 \leqslant \sum \sqrt{\alpha} \leqslant \sum \alpha \equiv \Omega(n),$$

the required result follows on using the theorem of Hardy and Ramanujan,§ that the number of integers n (not exceeding x) for which

$$|f(n) - \log\log n| > (\log\log n)^{\frac{1}{2}+\delta},$$

is $o(x)$ as x tends to infinity, for every $\delta > 0$, if $f(n) \equiv \omega(n)$ or $f(n) \equiv \Omega(n)$.

In the other direction there is clearly no corresponding result, for we have seen that $a(n) = 1$ with a positive asymptotic frequency.

† The first statement of (13) with the correct value of C appears to be that of P. Erdős, 'On an Elementary Proof of some Asymptotic Formulas in the Theory of Partitions', *Annals of Math.*, 43 (1942), 437. His method is different from that of Hardy, but is equally elementary.

‡ G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1945, 2nd ed.), 354.

§ See, for example, Hardy and Wright, loc. cit., Theorem 436.

A CLASS OF WEIERSTRASS BASES

By A. WINTNER (*Baltimore*)

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TOEPLITZ's '*D*-matrix'† belonging to a sequence $a_1, a_2,\ldots$, or to the corresponding ordinary Dirichlet series

$$A(s) = \sum_{m=1}^{\infty} a_m/m^s, \tag{1}$$

is the matrix of the linear substitution

$$\sum_{m=1}^{\infty} a_m x_{nm} = y_n \quad (n = 1, 2,\ldots). \tag{2}$$

He has shown (ibid.) that (i) this matrix transforms every vector $(x_1, x_2,\ldots)$, satisfying

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty, \tag{3}$$

into a vector $(y_1, y_2,\ldots)$, satisfying

$$\sum_{n=1}^{\infty} |y_n|^2 < \infty \tag{4}$$

(with the understanding that

$$\sum_{m=1}^{\infty} |a_m|^2 < \infty \tag{5}$$

must be assumed, since otherwise $y_1, y_2,\ldots$ will not even exist for unspecified $x_1, x_2,\ldots$ with $|x_1|^2+|x_2|^2+\ldots < \infty$), if and only if the Dirichlet series (1) is convergent in the half-plane $\sigma > 0$ and represents there a bounded function,

$$\text{l.u.b.}_{\sigma>0} |A(s)| < \infty; \tag{6}$$

(ii) the correspondence between the Hilbert x-space and the Hilbert y-space is one-to-one by virtue of (2) if and only if the additional condition

$$\text{g.l.b.}_{\sigma>0} |A(s)| \neq 0 \tag{7}$$

is satisfied; finally

(iii) in the latter case, the unique inverse of (2) (within Hilbert's space) is

$$\sum_{m=1}^{\infty} b_m y_{nm} = x_n \quad (n = 1, 2,\ldots), \tag{8}$$

† Cf. Toeplitz (4).

where $b_1, b_2, \ldots$, or the corresponding Dirichlet series

$$B(s) = \sum_{m=1}^{\infty} b_m/m^s, \tag{9}$$

is defined by

$$A(s)B(s) = 1. \tag{10}$$

If (6) is replaced by the more stringent condition

$$\sum_{m=1}^{\infty} |a_m| < \infty, \tag{11}$$

then (1) becomes absolutely convergent at the point $s = 0$, and hence on the whole half-plane $\sigma \geqslant 0$; and so what results from the first of the three criteria of Toeplitz is just a manifestation of a general theorem of Schur.† On the other hand, what corresponds to (but is not contained in) the remaining two of three criteria is the fact that, if a Dirichlet series (1) satisfies (11) and (7), then

$$\sum_{m=1}^{\infty} |b_m| < \infty \tag{12}$$

holds in virtue of (10) and (9); a fact which can be proved directly but actually is just a manifestation of a general theorem of Pitt (**2**) and others concerning uniformly almost-periodic functions.

In the same way as (3) is the condition 'polar' to (5) (with reference to the set of all linear forms (2) or, for that matter, with reference to the first of them), so the condition 'polar' to (11) is‡

$$x_n = O(1) \quad (n \to \infty). \tag{13}$$

In addition, if (11) is assumed, the linear transformation (2) will not only *exist* for all vectors $(x_1, x_2, \ldots)$ satisfying (13) but will transform every such vector into a vector $(y_1, y_2, \ldots)$ satisfying

$$y_n = O(1) \quad (n \to \infty). \tag{14}$$

All of this suggests that the considerations of (**5**) concerning approximations in the mean (L^2) can be modified so as to supply criteria for 'Weierstrass' approximations. What can be proved in this direction is contained in the following theorem:

If $\phi(t)$ is an even periodic function having an absolutely convergent Fourier series

$$\phi(t) = \sum_{n=1}^{\infty} a_n \cos nt \tag{15}$$

(where $a_0 = 0$ is no loss of generality), then every continuous function on

† Schur (**3**), 6.

‡ Cf. (**1**), 1443–8 where references are given to papers by Dixon and Helly.

the interval $0 \leqslant t \leqslant \pi$ is the uniform limit of finite linear combinations of the functions

$$1, \phi(t), \phi(2t),..., \phi(nt),... \tag{16}$$

whenever the coefficients of (15) *satisfy the following condition*:

The only bounded solution, $x_n = O(1)$, of the homogeneous Toeplitz equations

$$\sum_{m=1}^{\infty} a_m x_{nm} = 0 \quad (n = 1, 2,...) \tag{17}$$

is the trivial solution

$$x_1 = x_2 = ... = 0. \tag{18}$$

In order that this condition be satisfied, it is necessary that

$$a_1 \neq 0, \tag{19}$$

and it is sufficient that the trigonometric series

$$\psi(t) = \sum_{n=1}^{\infty} b_n \cos nt, \tag{20}$$

where

$$a_1 b_1 = 1, \quad \textit{and} \quad \sum_{d|n} a_{n/d} b_d = 0 \textit{ if } n = 2, 3,..., \tag{21}$$

be an absolutely convergent Fourier series.

Let $\phi_1(t), \phi_2(t),...$ be a sequence of continuous functions having the property that every continuous function on the interval $a \leqslant t \leqslant b$ is the uniform limit of finite linear combinations of the functions $\phi_n(t)$. According to F. Riesz, this will be the case if (and only if) every function $\alpha(t)$ which is of bounded variation on the interval $a \leqslant t \leqslant b$ and satisfies the integral conditions

$$\int_a^b \phi_n(t)\, d\alpha(t) = 0 \quad (n = 1, 2,...)$$

is constant on its continuity points contained in the open interval $0 < t < \pi$.† Hence, it is clear that the sequence (16) will have the desired property if

$$\int_0^{\pi} \phi(nt)\, d \sum_{m=1}^{\infty} \frac{x_m}{m} \sin mt = 0 \tag{22}$$

is possible for $n = 1, 2,...$ only in the case (18), provided that the coefficients $x_1, x_2,...$ occurring in (22) are restricted to values for which the sequence $x_1/1, x_2/2,...$ becomes the Fourier sine-sequence of a function of bounded variation. On the other hand, from (11) and (15),

$$\phi(nt) = \sum_{m=1}^{\infty} a_m \cos nmt. \tag{23}$$

† Cf. (1), 1469–70.

But the uniqueness theorem of Fourier-Stieltjes series shows that (22) becomes equivalent to (17) in virtue of (23). Hence, it is sufficient to ascertain that (11) and (17) imply (18) under the proviso that $x_1/1$, $x_2/2$,... is the sequence of the Fourier sine-constants of a function of bounded variation. But this proviso is less general than the restriction (13). Since it is assumed that (17) and (13) imply (18), the first assertion of the theorem follows.

The second assertion of the theorem is obvious from the first. For, if (19) is denied, then $a_1 = 0$, and so

$$x_1 = 1 \neq 0, \quad x_2 = x_3 = \ldots = 0$$

becomes a solution of (17) and (13).

In order to prove the remaining assertion of the theorem, suppose that (19) is satisfied. Then the infinite system (21) of linear equations can be solved recursively:

$$b_1 = 1/a_1, \quad b_2 = -a_2/a_1^2, \quad \ldots, \quad b_n = \Omega_{n-1}(a_2,\ldots,a_n)/a_1^n, \quad \ldots,$$

where Ω_n is a universal polynomial in the n variables $a_2,\ldots, a_{n+1}$. Since (21) is symmetric in a and b, the substitution

$$(a_1, a_2,\ldots) \to (b_1, b_2,\ldots) \quad (a_1 \neq 0, b_1 \neq 0)$$

is involutory, and hence birational. It is also clear that (21) is equivalent to (10) in virtue of (1) and (9).

The remaining assertion of the theorem is that, if the assumption (12), which has not been used, is satisfied, then (17), (13), and (11) imply (18). In order to verify this implication, replace n in (17) by nk, where $k = 1, 2,\ldots$, and multiply the result by b_k. This gives

$$\sum_{k=1}^{\infty} b_k \sum_{m=1}^{\infty} a_m x_{mkn} = \sum_{k=1}^{\infty} b_k . 0 = 0$$

for every n. In view of (11), (12), (13), this can be rearranged into

$$\sum_{n=1}^{\infty} x_{mn} \sum_{d|m} b_{m/d} a_d = 0.$$

It follows therefore from (21) that x_n is 0, as required by (18). This completes the proof of the theorem.

As a simple application, the following theorem will now be deduced:

If $\Re\lambda > 1$ *and*

$$\phi(t) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^\lambda} \cos mt, \tag{24}$$

where $\chi(m)$ *is a bounded and completely multiplicative function of* m

(for instance, if $|\chi(m)| = 1$ for every m), then every continuous function on the interval $0 \leqslant t \leqslant \pi$ is the uniform limit of finite linear combinations of the functions (16).

The same holds if

$$a_m = \chi(m)/m^\lambda \quad (\mathscr{R}\lambda > 1) \tag{25}$$

in (24) *is replaced by*

$$a_m = \mu(m)/m^\lambda \quad (\mathscr{R}\lambda > 1), \tag{26}$$

where $\mu(m)$ is Möbius' factor.

It is understood that by a 'completely multiplicative' function $\chi(m)$ is meant any sequence $\chi(1), \chi(2),\ldots$ satisfying $\chi(nm) = \chi(n)\chi(m)$ and containing at least one non-vanishing element (so that $\chi(1) = 1$). Thus it is clear that $\chi(n)$ is bounded if and only if the inequality

$$|\chi(p)| \leqslant 1 \tag{27}$$

holds for *every* prime.

Clearly, (11) and (19) are satisfied by (25). Hence, it is sufficient to verify (12), which, in view of (9) and (10), means the absolute convergence of

$$1/A(s) = \sum_{n=1}^{\infty} b_n/n^s \tag{28}$$

at $s = 0$. But (1) and (25) show that Euler's factorization

$$A(s) = \prod_p \sum_{k=0}^{\infty} \{\chi(p)/p^{s+\lambda}\}^k \tag{29}$$

is valid for $\mathscr{R}(s+\lambda) > 1$; in fact, (27) is equivalent to

$$|\chi(n)| \leqslant 1. \tag{30}$$

Since $\mathscr{R}\lambda > 1$, it is now seen from (27) that both the product (29) and the series (28) are absolutely convergent at $s = 0$.

The treatment of the case (26) is even simpler since (29) must then be replaced by

$$A(s) = \prod_p \{1+\mu(p)/p^{s+\lambda}\} = 1/\zeta(s+\lambda),$$

and hence (28) by

$$1/A(s) = \zeta(s+\lambda);$$

so that $b_{.n} = m^{-\lambda}$.

The example

$$ma_m = 1 \quad \text{if} \quad m = 2^k \quad \text{and} \quad a_m = 0 \quad \text{if} \quad m \neq 2^k \quad (k = 0, 1, 2,\ldots), \tag{31}$$

which is almost of the type (25), leads to the following curiosity:

Every continuous function on the interval $0 \leqslant t \leqslant \pi$ *is the uniform limit of finite linear combinations of the functions* (16) *if* $\phi(t)$ *is the continuous function*

$$\sum_{k=0}^{\infty} 2^{-k} \cos 2^k t, \tag{32}$$

which is nowhere differentiable (*finitely*).

In fact, (32) is the case $a = \frac{1}{2}$, $b = 2$ of the Weierstrass example

$$\sum_{k=0}^{\infty} a^k \cos b^k t,$$

in which, according to Hardy, only

$$0 < a < 1 \leqslant ab$$

is needed. On the other hand, it is clear from (31) that (1) reduces to

$$A(s) = \sum_{k=0}^{\infty} 2^{-k}/2^{ks} = (1-2^{-s-1})^{-1};$$

so that, from (28),

$$b_1 = 1, \quad b_2 = -\tfrac{1}{2}, \quad b_3 = b_4 = \ldots = 0.$$

Hence, (12) is satisfied. This proves the assertion, since (11) is obvious for (32).

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ON THE POSITIVE VALUES OF LINEAR FORMS

By J. H. H. CHALK (*London*)

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1. Let $L_1,\ldots, L_n$ be n homogeneous linear forms in n variables $u_1,\ldots, u_n$ with real coefficients whose determinant Δ is not zero. Various results† are known which tell us that there exist integral values of $u_1,\ldots, u_n$, not all zero, for which

$$|L_1 \ldots L_n| \leqslant k_n |\Delta|,$$

where k_n is a number depending only on n. The best possible value of k_n for which this is true is known when $n = 2$ or 3, and there are estimates of k_n for general n. It is of interest to consider what can be proved when $u_1,\ldots, u_n$ are restricted to values for which

$$L_1 > 0, \quad L_2 > 0, \quad \ldots, \quad L_n > 0. \tag{1}$$

For $n = 2$, this question (together with other questions) was solved by Segre and Mahler.‡ I prove the following generalization of their result.

THEOREM 1. *There exist integral values of* $u_1,\ldots, u_n$ *for which* (1) *holds, and for which*

$$L_1 L_2 \ldots L_n \leqslant |\Delta|. \tag{2}$$

The sign of equality is essential, as may be seen from the simple example

$$L_1 = \lambda_1 u_1, \quad \ldots, \quad L_n = \lambda_n u_n,$$

where $\lambda_1,\ldots, \lambda_n$ are numbers whose product is Δ. However, the determination of *all* sets of linear forms for which the result is not true with strict inequality in (2) does not seem to be easy, though it is facilitated by the use of a theorem of Hajós.§ I enumerate these sets for $n = 2, 3, 4$, in Theorems 4 (Cor.), 5, 6. A general property of such forms is given in Theorem 3.

An interesting question is whether the result of Theorem 1 remains the best possible if the conditions (1) are replaced by the weaker conditions

$$L_1 \geqslant 0, \quad \ldots, \quad L_n \geqslant 0. \tag{3}$$

† For a recent result of this kind, and for references, see H. Davenport, *Proc. K. Akad. Wet. Amsterdam*, 49 (1946), 822–8.

‡ B. Segre, *Duke Math. J.* 12 (1945), 337–65 and K. Mahler, ibid. 367–71.

§ G. Hajós, *Math. Zeitschrift*, 47 (1941), 427–67, reviewed in *Math. Reviews*, 3 (1942), 302. Hajós proved Minkowski's conjecture on the 'boundary case' of his theorem on linear forms; see Koksma, *Diophantische Approximationen*, ch. 2, § 2.

That this is the case when $n = 2$ was proved by Segre and Mahler, but their proof depends upon the properties of a particular type of quadratic form and does not appear to be capable of extension to general n.

There is a similar problem for non-homogeneous linear forms, and the case $n = 2$ has been considered by Davenport and Heilbronn in a recent paper.† The following theorem is a generalization of their result.

THEOREM 2. *Let $L_1,..., L_n$ be n homogeneous linear forms in n variables $u_1,..., u_n$ with real coefficients whose determinant is not zero. If $c_1,..., c_n$ are any real numbers, then there exist integral values of $u_1,..., u_n$ for which*

$$L_1+c_1 > 0, \quad L_2+c_2 > 0, \quad ..., \quad L_n+c_n > 0 \tag{4}$$

and

$$(L_1+c_1)(L_2+c_2)...(L_n+c_n) \leqslant |\Delta|. \tag{5}$$

This result is 'best possible', as may be seen from the example,

$$L_1 = \lambda_1 u_1, \quad ..., \quad L_n = \lambda_n u_n; \qquad c_1 = \lambda_1 C_1, \quad ..., \quad c_n = \lambda_n C_n,$$

where $\lambda_1,..., \lambda_n$ are any numbers whose product is Δ, and $C_1,..., C_n$ are integers. Theorem 2 includes Theorem 1, by taking

$$c_1 = ... = c_n = 0,$$

and so it will suffice to prove Theorem 2. Normally, such results on non-homogeneous linear forms do not imply the corresponding results for homogeneous linear forms, because in the former case simultaneous zero values of the variables are admitted. However, in the present case when $c_1 = ... = c_n = 0$, zero values are excluded by (4).

The problem of enumerating the forms which do not satisfy (5) with strict inequality presents difficulties similar to those encountered in the homogeneous case. As they are more complex, the case $n = 2$ only is considered in detail. I prove, however, the general theorem,

THEOREM 3. *Unless the ratios of the coefficients in each of the linear forms $L_1,..., L_n$ are all rational, the results of Theorems* 1 *and* 2 *are valid with strict inequality in* (2) *and* (5).

Since $$L_1+c_1 \geqslant 0, \quad L_2+c_2 \geqslant 0, \quad ..., \quad L_n+c_n \geqslant 0 \tag{6}$$

is a weaker hypothesis than (4), the result in Theorem 2 remains

† H. Davenport and H. Heilbronn, *J. of London Math. Soc.* (in course of publication).

valid if (4) is replaced by (6), and that it cannot be improved is obvious from the trivial example

$$L_1+c_1 = u_1-\delta, \quad \ldots, \quad L_n+c_n = u_n-\delta$$

where δ is a small positive number. In this case

$$\prod_r (L_r+c_r) \geqslant (1-\delta)^n$$

whenever (6) holds. However, for forms $L_1,\ldots, L_n$ which do not represent zero for integral values of $u_1,\ldots, u_n$ not all zero, it would be desirable to establish the existence, or otherwise, of a constant $k < 1$ which depends only upon these forms (and not on $c_1,\ldots, c_n$) such that we may satisfy (6) and

$$(L_1+c_1)(L_2+c_2)\ldots(L_n+c_n) \leqslant k|\Delta|.$$

Davenport and Heilbronn (loc. cit.) showed that when $n = 2$, such a k does not exist.

Another question which arises is whether (2) has an infinity of solutions subject to (1), provided that we suppose that none of the forms $L_1,\ldots, L_n$ represents zero. For $n = 2$, this was proved to be the case by Segre.

I am indebted to Professor H. Davenport for valuable suggestions and discussions.

2. We require a simple lemma. Let ϵ be any number satisfying

$$1 > \epsilon \geqslant 0. \tag{7}$$

LEMMA 1. *Let $x_1,\ldots, x_n$ be any real numbers satisfying*

$$|x_i| < 1, \tag{8}$$

$$\prod_i (1-x_i) \geqslant 1-\epsilon, \qquad \prod_i (1+x_i) \geqslant 1-\epsilon \tag{9}$$

for $i = 1, 2,\ldots, n$. Then

$$|x_i| \leqslant \sqrt{(2\epsilon)}. \tag{10}$$

Proof. The product of the inequalities in (9) gives

$$\prod_i (1-x_i^2) \geqslant (1-\epsilon)^2$$

and by (8) this implies $x_i^2 \leqslant 1-(1-\epsilon)^2 \leqslant 2\epsilon$. Hence $|x_i| < \sqrt{(2\epsilon)}$.

3. Proof of Theorem 2

By homogeneity, we may assume that $|\Delta| = 1$. We define M to be the lower bound of $(L_1+c_1)\ldots(L_n+c_n)$ for integral values of $u_1,\ldots, u_n$ subject to the conditions (4). If $M = 0$, there is nothing to prove; hence we can suppose $M > 0$. For any arbitrarily small

number $\epsilon_0 > 0$, there exist integral values $u_1^*,\ldots, u_n^*$ of $u_1,\ldots, u_n$ satisfying (4) for which

$$(L_1^*+c_1)\ldots(L_n^*+c_n) = \frac{M}{1-\epsilon}, \quad \text{where} \quad 0 \leqslant \epsilon < \epsilon_0. \tag{11}$$

Put
$$x_1 = \frac{L_1-L_1^*}{L_1^*+c_1}, \quad \ldots, \quad x_n = \frac{L_n-L_n^*}{L_n^*+c_n}. \tag{12}$$

Then the points $(x_1,\ldots,x_n)$ in n-dimensional space which correspond to integral values of $u_1,\ldots, u_n$ form a lattice of determinant

$$\Delta^* = [(L_1^*+c_1)\ldots(L_n^*+c_n)]^{-1}.$$

From the definition of M and (11), this lattice possesses the property that, if
$$x_1+1 > 0, \quad \ldots, \quad x_n+1 > 0, \tag{13}$$
then
$$(x_1+1)\ldots(x_n+1) = \frac{(L_1+c_1)\ldots(L_n+c_n)}{(L_1^*+c_1)\ldots(L_n^*+c_n)} \geqslant \frac{M}{M/(1-\epsilon)} = 1-\epsilon. \tag{14}$$

I prove that there is no lattice point, other than the origin O, in the cube (8). Suppose that $(x_1,\ldots,x_n)$ were such a lattice point. Then the conditions (13) are satisfied by $(x_1,\ldots,x_n)$ and $(-x_1,\ldots,-x_n)$ so that, by (14),

(i) $(x_1+1)\ldots(x_n+1) \geqslant 1-\epsilon$,

(ii) $(-x_1+1)\ldots(-x_n+1) \geqslant 1-\epsilon$.

From the lemma it follows that

$$|x_i| \leqslant \sqrt{(2\epsilon)} \quad (i = 1, 2,\ldots, n). \tag{15}$$

This is impossible unless $x_i = 0$ for $i = 1, 2,\ldots, n$, since otherwise, for any integer m, $(mx_1,\ldots, mx_n)$ is a lattice point, and we could choose m so that this new lattice point satisfied (8) but not (15). Now, by Minkowski's theorem on linear forms,

$$\Delta^* \geqslant 1, \tag{16}$$

i.e.
$$(L_1^*+c_1)\ldots(L_n^*+c_n) \leqslant 1.$$

Hence $M \leqslant 1-\epsilon$ by (11), so that *either* $M < 1$ *or* $M = 1$, $\epsilon = 0$. In any case, there is a solution of (5) subject to the conditions (4). This proves Theorem 2.

4. The cases of equality

In these cases,
$$(L_1+c_1)\ldots(L_n+c_n) \geqslant 1 \tag{17}$$
whenever (4) holds. This means that $M \geqslant 1$ and from the previous argument we must have $M = 1$ and $\epsilon = 0$. Hence

$$\Delta^* = 1,$$

and (14) may be written

$$(x_1+1)\dots(x_n+1) \geqslant 1. \tag{14'}$$

Now $x_1,\dots, x_n$ are linear forms in the variables $u_1-u_1^*,\dots, u_n-u_n^*$ of determinant $\Delta^* = 1$ which possess the property that there is no non-trivial solution in integers of the inequalities (8). By the theorem of Hajós (loc. cit.) we know that these forms may be reduced, by a change of order and an integral unimodular substitution, to the forms

$$x_1 = v_1, \quad x_2 = a_{21}v_1+v_2, \quad \dots, \quad x_n = a_{n,1}v_1+\dots+a_{n,n-1}v_{n-1}+v_n, \tag{18}$$

where $v_1,\dots, v_n$ are derived from $u_1-u_1^*,\dots, u_n-u_n^*$ by an integral unimodular substitution.

It is at once apparent that not all such forms satisfy (14') whenever (13) holds. Further, if $x_1,\dots, x_n$ are linear forms of the type (18) which satisfy (14') whenever (13) holds, then the forms $x_1,\dots, x_r$ $(r < n)$ must have the same property with r in place of n. For, if this were not the case, we could satisfy

$$(x_1+1)\dots(x_r+1) < 1$$

with

$$x_1+1 > 0, \quad \dots, \quad x_r+1 > 0$$

and by a suitable successive choice of $v_{r+1},\dots, v_n$ we could also satisfy

$$0 < (x_{r+1}+1) \leqslant 1, \quad \dots, \quad 0 < (x_n+1) \leqslant 1,$$

so that

$$(x_1+1)\dots(x_n+1) < 1,$$

which contradicts (14').

In discussing the cases of equality we shall restrict ourselves mainly to the homogeneous case. For this we have the additional information that $c_1 = \dots = c_n = 0$. By (12), this implies that in the space $(x_1,\dots, x_n)$ the point $(1, 1,\dots, 1)$ is a lattice point. It is convenient to make a simple transformation. If we put

$$y_1 = x_1+1, \quad \dots, \quad y_n = x_n+1,$$

then the points $(y_1,\dots, y_n)$ in this space, corresponding to integral values of $u_1,\dots, u_n$, generate the same lattice. Now

$$y_1 = \frac{L_1}{L_1^*}, \quad \dots, \quad y_n = \frac{L_n}{L_n^*}$$

are linear forms in the variables $u_1,\dots, u_n$ of determinant ± 1, and, from (13) and (14'),

$$y_1 \dots y_n \geqslant 1 \tag{19}$$

whenever

$$y_1 > 0, \quad \dots, \quad y_n > 0. \tag{20}$$

Since there is no non-trivial solution of

$$|y_1| < 1, \quad \ldots, \quad |y_n| < 1$$

in integers, we can as before reduce $y_1, \ldots, y_n$ to the forms

$$y_1 = v_1, \quad y_2 = a_{21}v_1 + v_2, \quad \ldots, \quad y_n = a_{n,1}v_1 + \ldots + a_{n,n-1}v_{n-1} + v_n. \tag{21}$$

The solution of the problem for any particular n is effected by applications of the following lemma.

5. LEMMA 2. *For the inequality $u(\alpha u+v+\beta) \geqslant 1$ to be true for all integers u, v satisfying*

$$u > 0, \qquad \alpha u+v+\beta > 0, \tag{22}$$

it is necessary and sufficient that

$$\alpha \equiv -\beta \equiv \frac{m}{m+1} \pmod 1, \tag{23}$$

where $m \geqslant 0$ is an integer.

Proof (*Necessity*). Without loss of generality (by writing

$$v = v' + mu + n$$

where m, n are integers), we can suppose that $0 \leqslant \alpha < 1$, $0 \leqslant \beta < 1$. If $\alpha+\beta = 0$, then $\alpha = \beta = 0$. If $\alpha+\beta > 0$ so that $u = 1$, $v = 0$ satisfies (22), then $\alpha+\beta \geqslant 1$. If $\alpha+\beta > 1$ so that $u = 1$, $v = -1$ also satisfies (22), then $\alpha+\beta-1 \geqslant 1$, which is impossible. Hence $\alpha+\beta = 1$, and the product may be written

$$u(\alpha u+v+1-\alpha).$$

There exists an integer $m \geqslant 0$ such that

$$\frac{m}{m+1} \leqslant \alpha < \frac{m+1}{m+2}.$$

We take $u = m+2$, $v = -(m+1)$ and have

$$u(\alpha u+v+1-\alpha) = (m+2)[(m+1)\alpha - m] < 1.$$

Hence

$$(m+1)\alpha - m \leqslant 0,$$

whence $\alpha = \dfrac{m}{m+1}$. This proves the necessity of the conditions (23).

(*Sufficiency*). It suffices to prove that

$$u\left\{\frac{m}{m+1}u+v+1-\frac{m}{m+1}\right\} \geqslant 1$$

for all integers u, v for which both factors are positive, and it involves

no loss of generality to consider values of u satisfying $1 \leqslant u < m+1$. Writing $u = 1+u'$,

$$u\left\{\frac{m}{m+1}u+v+1-\frac{m}{m+1}\right\} = (1+u')\left\{u'+v+1-\frac{u'}{m+1}\right\}$$

$$\geqslant (1+u')\left(1-\frac{u'}{m+1}\right) = 1+\frac{u'(m-u')}{m+1} \geqslant 1.$$

This proves the lemma.

6. From § 4, we have for the cases of equality (apart from the order of the forms)

$$\frac{L_1+c_1}{L_1^*+c_1} = v_1+1, \qquad \frac{L_2+c_2}{L_2^*+c_2} = a_{21}v_1+v_2+1, \quad ...,$$

$$\frac{L_n+c_n}{L_n^*+c_n} = a_{n,1}v_1+...+v_n+1,$$

where $v_1,..., v_n$ are derived from $u_1-u_1^*,..., u_n-u_n^*$ by an integral unimodular substitution. Hence to prove Theorem 3 it suffices to prove that $a_{21},..., a_{n,1},..., a_{n,n-1}$ are rational numbers.

LEMMA 3. *Let*

$$x_1 = v_1, \quad x_2 = a_{21}v_1+v_2, \quad ..., \quad x_n = a_{n,1}v_1+...+a_{n,n-1}v_{n-1}+v_n \tag{18}$$

be n linear forms in the variables $v_1,..., v_n$. If they satisfy

$$(x_1+1)...(x_n+1) \geqslant 1 \tag{14'}$$

whenever the integers $v_1,..., v_n$ are such that

$$x_1+1 > 0, \quad ..., \quad x_n+1 > 0, \tag{13}$$

then a_{rs} $(s < r \leqslant n)$ are rational numbers.

Proof. We assume that the lemma is true in n dimensions and prove that it is then true in $n+1$ dimensions. Since it is true in 2 dimensions, by Lemma 2, the result follows by induction on n.

By making a substitution of the form $v_r = l_{r,1}v_1+...+l_{r,r-1}v_{r-1}$ $(2 \leqslant r \leqslant n+1)$ where l_{rs} $(s < r \leqslant n+1)$ are suitable integers, we can suppose without loss of generality that

$$0 \leqslant a_{rc} < 1 \quad (s < r \leqslant n+1).$$

We distinguish two cases according as $a_{21} = 0$ or as $a_{21} \neq 0$.

Case 1: $a_{21} = 0$. We have now

$$x_1 = v_1, \quad x_2 = v_2, \quad x_3 = a_{31}v_1+a_{32}v_2+v_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_{n+1} = a_{n+1,1}v_1+...+v_{n+1}. \tag{24}$$

On putting $v_1 = 0$, we have $x_1+1 = 1$ and $x_2,\ldots, x_{n+1}$ satisfy the hypothesis (14′) whenever (13) holds. Hence, by the inductive hypothesis, all the coefficients in (24) except $a_{31}, a_{41},\ldots, a_{n+1,1}$ are rational numbers. Similarly, on putting $v_2 = 0$, all coefficients except $a_{32}, a_{42},\ldots, a_{n+1,2}$ are rational numbers. Thus every coefficient a_{rs} $(s < r \leqslant n+1)$ must be rational.

Case 2: $a_{21} \neq 0$. We now consider the forms

$$x_1 = v_1, \quad x_2 = a_{21}v_1+v_2, \quad \ldots,$$

$$x_{n+1} = a_{n+1,1}v_1+\ldots+a_{n+1,n}v_n+v_{n+1}. \tag{25}$$

From the remarks in § 4, it is plain that for $2 \leqslant r \leqslant n+1$ these forms satisfy

$$(x_1+1)\ldots(x_r+1) \geqslant 1 \tag{26}$$

whenever the integers $v_1,\ldots, v_r$ are such that

$$x_1+1 > 0, \quad \ldots, \quad x_r+1 > 0. \tag{27}$$

In particular, when $r = n$, we may apply the hypothesis to $x_1,\ldots, x_n$ and deduce that a_{rs} $(s < r \leqslant n)$ are rational numbers. On putting $v_1 = 0$, we have $x_1+1 = 1$ and $x_2,\ldots, x_{n+1}$ satisfy the hypothesis. Thus, in particular, $a_{n+1,2},\ldots, a_{n+1,n}$ are rational. It remains to prove that $a_{n+1,1}$ is rational.

On putting $r = 2$ in (26) and (27) and using Lemma 2, we have $a_{21} = m/(m+1)$ where $m > 0$ is an integer. When $v_1 = -v_2 = m$,

$$(x_1+1)(x_2+1) = (v_1+1)\left[\frac{m}{m+1}v_1+v_2+1\right] = 1.$$

By choosing $v_3,\ldots, v_{n+1}$ $(= v_3',\ldots, v_{n+1}')$ successively we can always ensure that

$$0 < (x_3+1) \leqslant 1, \quad \ldots, \quad 0 < (x_{n+1}+1) \leqslant 1.$$

But, since $(x_1+1)\ldots(x_{n+1}+1) \geqslant 1$, we must have for these values of $v_1,\ldots, v_{n+1}$,

$$x_3+1 = \ldots = x_{n+1}+1 = 1.$$

In particular,

$$x_{n+1}+1 = a_{n+1,1}m+a_{n+1,2}(-m)+a_{n+1,3}v_3'+\ldots+v_{n+1}'+1 = 1.$$

Since $a_{n+1,2},\ldots, a_{n+1,n}$ are rational, it follows that $a_{n+1,n}$ is rational. This proves the lemma, and completes the proof of Theorem 3.

7. THEOREM 4. *Let $n = 2$. Then all the cases for which the result of Theorem 2 is not valid with strict inequality are given by*

$$L_1 = \lambda_1 V_1, \qquad c_1 = \lambda_1(1+C_1);$$

$$L_2 = \lambda_2\left[\frac{m}{m+1}V_1+V_2\right], \qquad c_2 = \lambda_2\left[\frac{m}{m+1}C_1+C_2+1\right]; \tag{28}$$

and

$$L_1 = \lambda_1\left[\frac{m}{m+1}V_1+V_2\right], \qquad c_1 = \lambda_1\left[\frac{m}{m+1}C_1+C_2+1\right];$$

$$L_2 = \lambda_2 V_1, \qquad c_2 = \lambda_2(1+C_1), \tag{29}$$

where V_1, V_2 are linear integral forms in u_1, u_2 of determinant 1 (or -1), λ_1, λ_2 are positive numbers satisfying $\lambda_1\lambda_2 = |\Delta|$, and C_1, C_2 ($m \geqslant 0$) are integers.

Proof. We may assume that $|\Delta| = 1$ and proceed as in § 4. In this way we derive two linear forms x_1, x_2 which satisfy the conditions (13), (14′), and (18) for $n = 2$. Thus

$$x_1 = v_1, \qquad x_2 = a_{21}v_1+v_2$$

where v_1, v_2 are linear integral forms in $u_1-u_1^*, u_2-u_2^*$ of determinant 1 (or -1). We may suppose that $0 \leqslant a_{21} < 1$. Since these forms satisfy (14′) whenever (13) holds, it follows from Lemma 2 that $a_{21} = m/(m+1)$ where $m \geqslant 0$ is an integer. Thus, either

$$L_1+c_1 = (L_1^*+c_1)(v_1+1), \qquad L_2+c_2 = (L_2^*+c_2)\left(\frac{m}{m+1}v_1+v_2+1\right), \tag{30}$$

or

$$L_1+c_1 = (L_1^*+c_1)\left(\frac{m}{m+1}v_1+v_2+1\right), \qquad L_2+c_2 = (L_2^*+c_2)(v_1+1). \tag{31}$$

If we write $\lambda_1 = L_1^*+c_1, \lambda_2 = L_2^*+c_2$, then λ_1, λ_2 are positive numbers whose product is 1. Since v_1, v_2 are linear integral forms in $u_1-u_1^*$, $u_2-u_2^*$ of determinant 1 (or -1), we may write $v_1 = V_1+C_1$, $v_2 = V_2+C_2$ where V_1, V_2 are linear forms in u_1, u_2 of determinant 1 (or -1), and C_1, C_2 are integers. On substituting for v_1, v_2 in (30) and (31) we obtain the forms given by (28) and (29). Conversely, these forms have the property in question, by Lemma 2.

COROLLARY. *Let $n = 2$. All the cases for which the result of Theorem 1 is not valid with strict inequality are given by*

$$L_1 = \lambda_1 V_1, \qquad L_2 = \lambda_2 V_2. \tag{32}$$

For the homogeneous case we have $c_1 = c_2 = 0$. This is only possible in (28) or (29) when $m = 0$. This gives the result.

8. As was proved in § 4, the cases for which the sign of equality is necessary in Theorem 1, are certainly of the form

$$L_1 = \lambda_1 y_1, \quad \ldots, \quad L_n = \lambda_n y_n \tag{33}$$

where $\lambda_1, \ldots, \lambda_n$ are positive numbers whose product is $|\Delta|$; and $y_1, \ldots, y_n$ can be transformed by a change of order and an integral unimodular substitution into

$$y_1 = v_1, \quad y_2 = v_2, \quad y_3 = a_{31}v_1 + a_{32}v_2 + v_3, \quad \ldots,$$
$$y_n = a_{n,1}v_1 + \ldots + a_{n,n-1}v_{n-1} + v_n. \tag{34}$$

It is both necessary and sufficient that they satisfy the conditions that

(i) $$y_1 \ldots y_n \geqslant 1$$

for all integer values of $v_1, \ldots, v_n$, which satisfy (20);

(ii) there exist integer values of $v_1, \ldots, v_n$ for which

$$y_1 = \ldots = y_n = 1.$$

As a consequence of (i) we have $y_1 \ldots y_r \geqslant 1$ $(r = 2, 3, \ldots, n-1)$ for all integer values of $v_1, \ldots, v_r$ which satisfy the conditions (20) with r in place of n. Without loss of generality, we may take $0 \leqslant a_{rs} < 1$.

9. In the case $n = 3$, we have

$$y_1 y_2 y_3 = v_1 v_2 (a_{31} v_1 + a_{32} v_2 + v_3). \tag{35}$$

With $v_1 = 1$ we deduce from Lemma 2 that

either $$a_{31} = \frac{1}{m+1}, \qquad a_{32} = \frac{m}{m+1},$$

or $$a_{31} = a_{32} = 0.$$

Similarly, with $v_2 = 1$, we have

either $$a_{31} = \frac{m'}{m'+1}, \qquad a_{32} = \frac{1}{m'+1},$$

or $$a_{31} = a_{32} = 0.$$

Hence the only possibilities are

$$a_{31} = a_{32} = \tfrac{1}{2}, \quad \text{or} \quad a_{31} = a_{32} = 0.$$

Conversely, the corresponding products satisfy the conditions (i) and (ii) of § 8. This leads to the following theorem.

THEOREM 5. *Let $n = 3$. All linear forms L_1, L_2, L_3 for which the sign of equality is necessary in Theorem* 1 *are given (after a possible change of order) by*

$$L_1 = \lambda_1 v_1, \qquad L_2 = \lambda_2 v_2, \qquad L_3 = \lambda_3 v_3, \tag{36}$$

$$L_1 = \lambda_1 v_1, \qquad L_2 = \lambda_2 v_2, \qquad L_3 = \lambda_3[\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_3], \tag{37}$$

where v_1, v_2, v_3 are linear integral forms in u_1, u_2, u_3 of determinant 1 *(or -1) and λ_1, λ_2, λ_3 are positive numbers satisfying $\lambda_1\lambda_2\lambda_3 = |\Delta|$.*

10. For the case $n = 4$, we have two possibilities to consider. Either $y_1y_2y_3y_4$ is of the form

$$v_1v_2v_3(a_{41}v_1+a_{42}v_2+a_{43}v_3+v_4) \quad \text{(case 1)}, \tag{38}$$

or of the form

$$v_1v_2(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_3)(a_{41}v_1+a_{42}v_2+a_{43}v_3+v_4) \quad \text{(case 2)}. \tag{39}$$

Case 1. We may take $v_2 = v_3 = 1$, so that

$$y_1y_2y_3y_4 = v_1(a_{41}v_1+a_{42}+a_{43}+v_4).$$

From Lemma 2, it follows that

$$a_{41} = \frac{m_1}{m_1+1}, \qquad a_{42}+a_{43} \equiv -\frac{m_1}{m_1+1} \pmod 1, \quad m_1 \geqslant 0 \text{ an integer}. \tag{40}$$

Similarly, by taking $v_1 = v_3 = 1$ and $v_1 = v_2 = 1$, we obtain

$$a_{42} = \frac{m_2}{m_2+1}, \qquad a_{41}+a_{43} \equiv -\frac{m_2}{m_2+1} \pmod 1, \quad m_2 \geqslant 0 \text{ an integer}. \tag{41}$$

$$a_{43} = \frac{m_3}{m_3+1}, \qquad a_{41}+a_{42} \equiv -\frac{m_3}{m_3+1} \pmod 1, \quad m_3 \geqslant 0 \text{ an integer}. \tag{42}$$

These conditions on a_{41}, a_{42}, a_{43} reduce to

$$a_{41} = \frac{m_1}{m_1+1}, \qquad a_{42} = \frac{m_2}{m_2+1}, \qquad a_{43} = \frac{m_3}{m_3+1}, \tag{43}$$

where $m_1 \geqslant 0$, $m_2 \geqslant 0$, $m_3 \geqslant 0$ are integers satisfying

$$\frac{1}{m_1+1}+\frac{1}{m_2+1}+\frac{1}{m_3+1} \equiv 0 \pmod 1. \tag{44}$$

The solutions of (44) are easily obtained, and, apart from order, are given by

$$(m_1, m_2, m_3) = (0,0,0), (1,1,0), (2,2,2), (3,3,1), (5,2,1). \tag{45}$$

These give the following possibilities for the product $y_1y_2y_3y_4$, no two of which are equivalent:

$$(a)\quad v_1v_2v_3v_4,$$

$$(b)\quad v_1v_2v_3(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_4),$$

$$(c)\quad v_1v_2v_3(\tfrac{2}{3}v_1+\tfrac{2}{3}v_2+\tfrac{2}{3}v_3+v_4),$$

$$(d)\quad v_1v_2v_3(\tfrac{1}{2}v_1+\tfrac{3}{4}v_2+\tfrac{3}{4}v_3+v_4),$$

$$(e)\quad v_1v_2v_3(\tfrac{5}{6}v_1+\tfrac{2}{3}v_2+\tfrac{1}{2}v_3+v_4).$$

Of these, the products (a), (b), (c), and (d) satisfy the conditions (i) and (ii) of paragraph 8. The product (e) is inadmissible, as may be seen by taking $v_1 = 1$, $v_2 = 2$, $v_3 = 2$, $v_4 = -3$.

Case 2. If we put $v_1 = v_2 = 1$, then

$$\begin{aligned} y_1y_2y_3y_4 &= (1+v_3)(a_{41}+a_{42}+a_{43}v_3+v_4) \\ &= v_3'(a_{41}+a_{42}-a_{43}+a_{43}v_3'+v_4). \end{aligned}$$

By Lemma 2,

$$a_{43} = \frac{m_4}{m_4+1}, \qquad a_{41}+a_{42}-a_{43} \equiv -\frac{m_4}{m_4+1} \pmod 1. \tag{46}$$

From these, we have that

$$a_{41}+a_{42} \equiv 0 \pmod 1. \tag{47}$$

On taking $v_1 = 2$, $v_2 = 1$, $v_3 = -1$,

$$y_1y_2y_3y_4 = 2a_{41}+a_{42}-a_{43}+v_4.$$

Hence $2a_{41}+a_{42}-a_{43}$ must be an integer. This implies that

$$2a_{41}+a_{42} \equiv \frac{m_4}{m_4+1} \pmod 1. \tag{48}$$

Similarly, on taking $v_1 = 1$, $v_2 = 2$, $v_3 = -1$, we find that

$$a_{41}+2a_{42} \equiv \frac{m_4}{m_4+1} \pmod 1. \tag{49}$$

From (47), (48), (49) we deduce that

$$a_{41} = a_{42} = \frac{m_4}{m_4+1} \pmod 1, \tag{50}$$

and that

$$\frac{2m_4}{m_4+1} \equiv 0 \pmod 1. \tag{51}$$

Hence, $m_4 = 0$ or 1, and

either $\quad a_{41} = a_{42} = a_{43} = 0 \quad$ or $\quad a_{41} = a_{42} = a_{43} = \tfrac{1}{2}$. (52)

The first case gives

$$y_1y_2y_3y_4 = v_1v_2(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_3)v_4, \tag{53}$$

which (apart from order) is the same as the product (b) in Case 1. The other case gives

$$y_1y_2y_3y_4 = v_1v_2(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_3)(\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+\tfrac{1}{2}v_3+v_4), \tag{54}$$

which satisfies the conditions (i) and (ii) of § 8.

The results of these two cases give the following theorem.

THEOREM 6. *Let $n = 4$. All linear forms for which the sign of equality is necessary in Theorem* 1 *are given (apart from a change of order) by*

$$L_1 = \lambda_1 v_1, \quad L_2 = \lambda_2 v_2, \quad L_3 = \lambda_3 v_3, \quad L_4 = \lambda_4 v_4; \tag{55}$$

$$L_1 = \lambda_1 v_1, \quad L_2 = \lambda_2 v_2, \quad L_3 = \lambda_3 v_3,$$
$$L_4 = \lambda_4[\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_4]; \tag{56}$$

$$L_1 = \lambda_1 v_1, \quad L_2 = \lambda_2 v_2, \quad L_3 = \lambda_3 v_3,$$
$$L_4 = \lambda_4[\tfrac{2}{3}v_1+\tfrac{2}{3}v_2+\tfrac{2}{3}v_3+v_4]; \tag{57}$$

$$L_1 = \lambda_1 v_1, \quad L_2 = \lambda_2 v_2, \quad L_3 = \lambda_3 v_3,$$
$$L_4 = \lambda_4[\tfrac{1}{2}v_1+\tfrac{3}{4}v_2+\tfrac{3}{4}v_3+v_4]; \tag{58}$$

$$L_1 = \lambda_1 v_1, \quad L_2 = \lambda_2 v_2, \quad L_3 = \lambda_3[\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+v_3],$$
$$L_4 = \lambda_4[\tfrac{1}{2}v_1+\tfrac{1}{2}v_2+\tfrac{1}{2}v_3+v_4], \tag{59}$$

where v_1, v_2, v_3, v_4 are linear integral forms in u_1, u_2, u_3, u_4 of determinant 1 (*or* -1) *and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive numbers whose product is $|\Delta|$.*

ITERATION OF THE EXPONENTIAL FUNCTION

By E. M. WRIGHT (*Aberdeen*)

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1. THE problem of the iteration of an analytic function $f(x)$ is closely related to that of finding an analytic solution of the equation

$$\phi\{f(x)\} = \phi(x)+1 \tag{1}$$

due to Abel (1839) or of the equivalent Schroeder's equation

$$\psi\{f(x)\} = \alpha\psi(x). \tag{2}$$

S. Pincherle (1905) gave an account of work on these topics prior to 1905. Of the many later articles we may mention those by Fatou (1919), Julia (1918, 1931), Valiron (1931), and Hadamard (1944).

In the usual notation we write $f_n(x)$ for the nth iterate of $f(x)$. Let $x = \zeta$ be a root of the equation $f(x) = x$ and $a = f'(\zeta)$. Suppose first that $0 < |a| < 1$, so that ζ is an *attractive double point* of the substitution $\{x|f(x)\}$. Koenigs (1884, 1885) proved (i) that

$$y(x) = \lim_{n\to\infty} a^{-n}\{f_n(x)-\zeta\}$$

exists and is a regular function of x in a suitable region enclosing the point ζ, (ii) that $\psi(x) = y(x)$ is a solution of (2) for $\alpha = a$, (iii) that $y(\zeta) = 0$, $y'(\zeta) = 1$, and (iv) that any other solution of (2) regular in a region enclosing $x = \zeta$ is a constant multiple of a positive integral power of $y(x)$ with α the same power of a. If $|a| > 1$, so that ζ is a *repulsive double point* of the substitution $\{x|f(x)\}$, we take $y(x)$ to be the Koenigs function for the inverse of $f(x)$. In either case it follows from (ii) and (iii) that $y(x)$ has an inverse function $X(y)$ regular in a circle of positive radius with centre at $y = 0$ and such that

$$x = X(y), \qquad f(x) = X(ay),$$

and

$$X(ay) = f\{X(y)\}. \tag{3}$$

The rth iterate of $f(x)$ is given by

$$f_r(x) = X(a^r y)$$

for integral r, and we can use this to define $f_r(x)$ for non-integral r. The cases in which $a = 0$ or $|a| = 1$ have also been dealt with; a short account of the corresponding results is given by Julia (1931).

Further development of these results enabled Koenigs and others

to study the analytic theory of iteration in considerable detail and to solve a variety of functional equations. Attention has largely been concentrated, however, on rational $f(x)$. Here, after a brief discussion of the properties of $X(y)$ when $f(x)$ is an integral function, I find the power-series expansion of $X(y)$ when $f(x) = ce^x$ $(c > 0)$. By (3), $X(y)$ is an analytic solution of the equation

$$X(\zeta y) = ce^{X(y)} \quad (c > 0), \tag{4}$$

where ζ is a root of $x = ce^x$. I also describe briefly the Riemann surface of $y(x)$ when ζ is a repulsive double point of the substitution $(x|ce^x)$.

I am indebted to Dr. A. J. Macintyre for many references to the literature and to Professor A. G. Walker for an inquiry which drew my attention to the problem. Walker (1946) gives references to the work of M'Crea, Milne, Walker, and Whitrow on the problem of the iteration of real, continuous and steadily increasing (but not necessarily analytic) functions of a real variable. P. Lévy (1928) applied his theory of functions of regular growth to this problem with great success and discussed (Lévy 1927) the iteration of the exponential from this point of view. Since the present note is concerned with the analytic problem, there is no overlap between Lévy's work and mine.

2. I prove first two simple theorems about $y(x)$ and its inverse $X(y)$ when $f(x)$ is any integral function. Fatou (1919, 48, 281–3) proved corresponding results for rational $f(x)$.

THEOREM 1. *If $f(x)$ is an integral function, if $f(\zeta) = \zeta$, and if $|f'(\zeta)| > 1$, then the Koenigs function $y(x)$ of $f(x)$ with respect to ζ is the inverse of an integral function $X(y)$. If $f(x)$ has the exceptional value A, so has $X(y)$. If $f'(x)$ has no zero, neither has $X'(y)$, and $y(x)$ has no algebraic singularities.*

Let $a = f'(\zeta)$, so that $|a| > 1$. By Koenigs' results, $X(y)$ is regular for $|y| < R$ for some $R > 0$. Since $f(x)$ is an integral function, it follows by repeated use of (3) that $X(a^n y)$ is regular for $|y| < R$ and every positive integral n. But $|a| > 1$ and so $X(y)$ is regular for every finite y.

By (3), if $f(x) \neq A$ for any x, $X(ay) \neq A$ for any y. Differentiating (3) with respect to y and putting y/a for y, we have

$$aX'(y) = f'\{X(y/a)\}X'(y/a).$$

If $f'(x)$ never vanishes and $X'(y_0) = 0$, it follows from this that

$$X'(y_0 a^{-1}) = X'(y_0 a^{-2}) = \ldots = X'(y_0 a^{-n}) = 0.$$

This is impossible, for $y_0 a^{-n} \to 0$ as $n \to \infty$ and $X(y)$ is regular and not constant at and near $y = 0$.

If we replace $f(x)$ by its inverse in Theorem 1, we have as an immediate corollary

THEOREM 2. *If $f(x)$ is the inverse of an integral function, if $f(\zeta) = \zeta$, and if $0 < |f'(\zeta)| < 1$, then $y(x)$ is the inverse of an integral function $X(y)$. If the inverse of $f(x)$ has the exceptional value A, so has $X(y)$. If $f(x)$ has no algebraic singularities, neither has $y(x)$.*

3. For the rest of this paper I take $f(x) = ce^x$, where $c > 0$. A double point ζ of the substitution $(x|ce^x)$ is attractive or repulsive according as $|\zeta| \lesseqgtr 1$. The following lemma contains all we require about these double points; the results are due to Lémeray (1896, 1897).

LEMMA. *If $c > 0$, the equation $x = ce^x$ has complex roots $\xi_p \pm i\eta_p$ $(p = 1, 2, \ldots)$ such that $2p\pi < \eta_p < (2p+1)\pi$. The only other roots of the equation are*

(i) *if $c > e^{-1}$, two extra complex roots corresponding to $p = 0$ above;*

(ii) *if $c = e^{-1}$, a double root at $x = 1$;*

(iii) *if $c < e^{-1}$, two real roots ξ, ξ' such that*

$$0 < \xi < 1 < \log(1/c) < \xi'.$$

All these roots have their moduli greater than unity, and so are repulsive double points of the substitution $(x|ce^x)$, except the root 1 in case (ii) *and the root ξ in case* (iii). *This ξ is an attractive double point of the substitution.*

By Koenigs' results, $X(y)$ is a regular function of y in a circle of positive radius surrounding the origin. We may therefore write

$$X(y) = \sum_{m=0}^{\infty} a_m y^m \tag{5}$$

and we know that $a_0 = \zeta$, $a_1 = 1$. I shall prove

THEOREM 3. *If $f(x) = ce^x$ and if ζ is a repulsive double point of the substitution $(x|ce^x)$, then $X(y)$ is an integral function and*

$$ma_m(1-\zeta^{1-m}) = \sum_{p=1}^{m-1} pa_p a_{m-p} \zeta^{-p} \quad (m \geqslant 2). \tag{6}$$

The first conclusion follows from Theorem 1. To obtain (6) we differentiate (4) logarithmically and have

$$\zeta X'(\zeta y) = X'(y)X(\zeta y).$$

Substituting from (5) and equating coefficients of y^{m-1}, we have

$$ma_m\zeta^m = \sum_{p=1}^{m} pa_p a_{m-p}\zeta^{m-p} \quad (m \geqslant 1).$$

When $m = 1$, this is an identity. When $m \geqslant 2$, we put $a_0 = \zeta$ and deduce (6).

We observe that (6) gives us in succession a_2, a_3,... in terms of earlier members of the $\{a_m\}$ sequence. The similarity of the recurrence relation (6) to one already discussed (Wright 1945), viz.

$$c_1 = 1, \quad (m-1)c_m = \sum_{p=1}^{m-1} c_p c_{m-p} e^{-(p-1)\delta} \quad (m \geqslant 2; \delta > 0),$$

is apparent. The function $j(m)$ defined in my earlier note gives a convenient measure of the size of a_m much as it did for c_m. We recall that $j(t)$ is defined as a function of a continuous variable $t \geqslant 1$ by the relations

$$j(t) = 0 \quad (1 \leqslant t < e), \qquad j(t) = j(\log t)+1 \quad (t \geqslant e).$$

The sequence $\{e_h\}$ is defined by

$$e_0 = 1, \qquad e_h = \exp(e_{h-1}) \quad (h \geqslant 1),$$

so that $\quad e_{j(t)} \leqslant t < e_{j(t)+1}, \qquad (\log)^{j(t)+1} t < 1 < (\log)^{j(t)} t.$

I shall prove

THEOREM 4. *If ζ is a repulsive double point, then*

$$|a_m| < K^m |\zeta|^{-mj(m)}$$

for some number K independent of m.

We define the sequence $\{b_m\}$ by

$$b_0 = |\zeta|, \quad b_1 = 1, \quad mb_m(1-|\zeta|^{1-m}) = \sum_{p=1}^{m-1} pb_p b_{m-p} |\zeta|^{-p}$$

and write $Z(y) = \sum_{m=0}^{\infty} b_m y^m$. It follows from (6) by induction that $|a_m| \leqslant b_m$. Since b_m is the same function of $|\zeta|$ that a_m is of ζ, the function $Z(y)$ is integral and

$$Z(|\zeta|y) = Ce^{Z(y)}, \quad C = |\zeta|e^{-|\zeta|} \leqslant e^{-1} < 1.$$

Hence $$|Z(|\zeta|y)| < e^{|Z(y)|}.$$

If h is so chosen that

$$\max_{|y|\leqslant 1} |Z(y)| \leqslant e_h,$$

it follows that $|Z(y)| \leqslant e_{h+l}$ for $|y| \leqslant |\zeta|^l$ and so

$$b_m = \frac{1}{2\pi i} \int_{|y|=|\zeta|^l} \frac{Z(y)}{y^{m+1}} dy \leqslant \frac{e_{h+l}}{|\zeta|^{ml}}.$$

If we choose $l = j(m)-h+1$ and take m sufficiently large to make $l > 0$, we have finally

$$b_m \leqslant e_{j(m)+1} |\zeta|^{-mj(m)+m(h-1)} < K^m |\zeta|^{-mj(m)}.$$

4. If ζ is a repulsive double point of the substitution $(x|ce^x)$, the functions $X(y)$ and $X'(y)$ never vanish, by Theorem 1. Hence the Riemann surface of $y(x)$ has no algebraic branch points but a transcendental branch point at $x = 0$ in every sheet. If the point x follows a path encircling the origin, ce^x follows a path encircling the point c; since $y(ce^x) = \zeta y(x)$, this implies that c is also a branch point (though not in all sheets) and similarly for all the consequents of $x = 0$, viz. c, ce^c, ce^{ce^c},.... If $c < e^{-1}$, these consequents have a limit point at $x = \xi$, where ξ is the real attractive double point mentioned in the lemma.

The other double points ζ', ζ'',... do not appear to be singularities of $y(x)$. The equation

$$y(ce^x) = \zeta y(x)$$

holds in the neighbourhood of $x = \zeta'$, but the two y-functions refer to different branches. Hence there is no contradiction of Koenigs' theorem that all solutions of (2) regular at $x = \zeta'$ are integral powers of the $y(x)$ associated with ζ' and have $\alpha = \zeta'^n$, for in this theorem it is implied that $\psi(x)$ is uniform at and near $x = \zeta'$.

5. We now suppose $c < e^{-1}$ so that, by our lemma, the substitution $(x|ce^x)$ has two real double points ξ, ξ', where $0 < \xi < 1 < \xi'$. Of these ξ is attractive and ξ' repulsive. It may readily be proved, either directly or by means of Schwarz's lemma, that, under repetitions of the substitution, the transforms of any point in the half-plane $\mathscr{R}(x) \leqslant \xi'$ (except ξ' itself) tend to ξ. (This is in fact true of a more extended region). By Koenigs (1884), the function $y(x)$ associated with ξ is therefore regular for $\mathscr{R}(x) \leqslant \xi'$, except at ξ' where $y(x)$ has a singularity.

The proof of the following theorem is obvious from that of Theorem 3; in fact (7) is the same as (6) with ξ for ζ.

THEOREM 5. *If $0 < c < e^{-1}$ and $X(y)$ is associated with the attractive double point ξ of the substitution $(x|ce^x)$, then $a_0 = \xi$, $a_1 = 1$, and*

$$-ma_m(1-\xi^{m-1}) = \sum_{p=1}^{m-1} pa_p a_{m-p} \xi^{m-p-1} \quad (m \geqslant 2). \tag{7}$$

The function (7) differs from (6) only in the substitution of ξ for ζ, but this difference is substantial, since $\xi < 1$ and $|\zeta| > 1$. The behaviour of a_m for large m is now entirely different, as is shown by the next theorem, from which it also follows that $X(y)$ is no longer an integral function. The distinction between the two sorts of recurrence relation typified by (6) and (7) has been discussed by Cooper (1947).

THEOREM 6. *If*

$$D = \frac{1}{2}\left\{1 + \sqrt{\frac{1+\xi}{1-\xi}}\right\},$$

then

$$\frac{D^{m-1}}{m} \leqslant (-1)^{m-1}a_m \leqslant \frac{1}{(1-\xi)^{m-1}} \quad (m \geqslant 1).$$

We write $b_m = (-1)^{m-1}a_m$ so that (7) becomes

$$mb_m(1-\xi^{m-1}) = \sum_{p=1}^{m-1} pb_p b_{m-p} \xi^{m-p-1}. \tag{8}$$

Since $b_1 = 1$, we see that $b_m > 0$ for $m \geqslant 1$.

We use induction to prove both parts of Theorem 6. The inequality $b_m \leqslant (1-\xi)^{1-m}$ is trivial for $m = 1$; let us suppose it true for $m \leqslant M-1$. By (8),

$$\begin{aligned} Mb_M(1-\xi^{M-1}) &\leqslant M(1-\xi)^{2-M} \sum_{p=1}^{M-1} \xi^{M-p-1} \\ &= M(1-\xi)^{1-M}(1-\xi^{M-1}). \end{aligned}$$

Hence, for $M \geqslant 2$, $b_M \leqslant (1-\xi)^{1-M}$.

Again, $mb_m \geqslant D^{m-1}$ for $m = 1$ and $m = 2$, the latter since

$$4b_2 = \frac{2}{1-\xi} = 1 + \frac{1+\xi}{1-\xi} > 1 + \sqrt{\frac{1+\xi}{1-\xi}} = 2D.$$

Let us suppose that $mb_m \geqslant D^{m-1}$ for $m \leqslant M-1$, where $M \geqslant 3$. By (8),

$$\begin{aligned} Mb_M > Mb_M(1-\xi^{M-1}) &\geqslant (M-1)b_{M-1}b_1 + (M-2)b_{M-2}b_2\xi \\ &\geqslant D^{M-2} + D^{M-3}\frac{\xi}{2(1-\xi)} = D^{M-1}, \end{aligned}$$

since $$\frac{\xi}{2(1-\xi)} = \frac{1}{4}\left(\frac{1+\xi}{1-\xi}-1\right) = D(D-1) = D^2-D.$$

This completes the proof of Theorem 6.

THEOREM 7. *The radius of convergence of the power-series for $X(y)$ is ρ, where*

$$1-\xi \leqslant \rho = -\xi^{-1}y(0) \leqslant \min\left\{\frac{1}{D}, \frac{1-e^{-D\xi}}{D\xi}\right\}.$$

$X(y)$ is regular at all points on the circumference of the circle of convergence, except at $y = -\rho$. The function

$$X(y)-\log(y+\rho)$$

is regular in a circle $|y| < \rho+\delta$ for some $\delta > 0$.

It follows from Theorem 6 that

$$1-\xi \leqslant \rho \leqslant \frac{1}{D}.$$

We now write $z = -y$ and

$$F(z) = \xi - X(y) = \sum_{m=1}^{\infty} b_m z^m,$$

so that $$\xi - F(\xi z) = \xi e^{-F(z)} \tag{9}$$

by (4). Suppose z to increase from 0 through positive real values; since every $b_m > 0$, $F(z)$ will also increase through positive real values until z reaches a singularity of $F(z)$ at $z = \rho$. By (9), it is then obvious that $F(\xi\rho) = \xi$ and that $F(z)$ has a logarithmic singularity at ρ. If $|z| \leqslant \xi\rho$ and $z \neq \xi\rho$, then $|F(z)| < \xi$. Hence, by (9), $F(z)$ is regular at all points on the circumference $|z| = \rho$ except at $z = \rho$ itself.

Since $$F'(\xi\rho) = \sum_{m=1}^{\infty} m b_m \xi^{m-1}\rho^{m-1} > 0,$$

the left-hand side of (9) has a simple zero when $z = \rho$. Hence

$$F(z)+\log(\rho-z) = \log\left\{\frac{\xi(\rho-z)}{\xi - F(\xi z)}\right\}$$

is regular at $z = \rho$ and so

$$X(y)-\log(y+\rho)$$

is regular at all points of the circle $|y| \leqslant \rho$ and so in a circle of slightly greater radius.

Since $F(\xi\rho) = \xi$, it follows that $X(-\xi\rho) = 0$ and $-\xi\rho = y(0)$, for $y(x)$ is the inverse of $X(y)$. Finally, since $mb_m \geqslant D^{m-1}$,

$$D\xi = DF(\xi\rho) = \sum_{m=1}^{\infty} b_m D\xi^m \rho^m \geqslant -\log(1-D\xi\rho)$$

and so

$$\rho \leqslant \frac{1-e^{-D\xi}}{D\xi}.$$

This completes the proof of Theorem 7.

Numerical calculations show that

$$\frac{1-e^{-D\xi}}{D\xi} \gtrless \frac{1}{D}$$

according as $\xi \gtrless 0{\cdot}82\ldots$, that is, according as $c \gtrless 0{\cdot}36\ldots$.

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A THEOREM ON STAR DOMAINS

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I. Introduction

Let K be a bounded symmetrical star domain, or, shortly, a *star domain*,† in the (x, y)-plane, i.e. a bounded closed point-set of the following kind:

(i) The origin, $O = (0, 0)$, is an inner point of K;

(ii) If the point $P = (x, y)$ belongs to K, then so does

$$\overline{P} = (-x, -y);$$

(iii) The boundary C of K is a Jordan curve which meets every radius vector from O in just one point.

We shall always use a superscript bar, as $\overline{P}$ above, to indicate the image of a point, or point-set, in the origin.

The *lattice* Λ of *basis*

$$D = (x_1, y_1), \qquad E = (x_2, y_2),$$

and of *determinant*

$$d(\Lambda) = |x_1 y_2 - x_2 y_1| > 0,$$

consists of all points P of the form

$$P = uD + vE \quad (u, v = 0, \pm 1, \pm 2, \ldots).$$

In particular, $O = (0, 0)$ is a point of Λ. The lattice Λ is called *K-admissible* if no lattice-point, except O, is an inner point of K. The lower bound,

$$\Delta(K) = \text{lim. inf. } d(\Lambda),$$

of $d(\Lambda)$, extended over all K-admissible lattices, is a finite positive number. We call Λ a *critical lattice* of K if it is K-admissible and

$$d(\Lambda) = \Delta(K).$$

It is easily seen that

$$\Delta(H) \leqslant \Delta(K)$$

if H is a star domain contained in K. We say that K is *irreducible* if the stronger inequality

$$\Delta(H) < \Delta(K)$$

holds for all H contained in, but distinct from, K.

† The definitions follow K. Mahler, *Proc. Kon. Nederland. Akad. Wet.* 50 (1947), 98–107, but the notation is rather different. I shall refer to this paper as M.

Finally, a star domain is *convex* if it contains, with any two points F and G, also all points

$$(1-t)F+tG \quad (0 \leqslant t \leqslant 1)$$

of the line segment FG joining the two points.

Mahler has shown† that, if K is an irreducible *convex* star domain which is not a parallelogram and if P is any point on the boundary C of K, there exists a unique critical lattice $\Lambda = \Lambda(P)$ containing P. The lattice $\Lambda(P)$ has just six points P, Q, $R = Q-P$, $\overline{P}$, $\overline{Q}$, and $\overline{R}$, on C, and, as P moves continuously round C in a positive direction, the other points of $\Lambda(P)$ on C do likewise. Conversely,‡ if K is any convex star domain, and there exists a constant δ such that, for each point P on C, there is a lattice $\Lambda(P)$ of determinant δ having at least two other points Q and $R = Q-P$ on C, then K is irreducible, and

$$\Delta(K) = \delta.$$

If K is irreducible but not convex, the situation is more complicated. If P is a point of C, there is still§ at least one critical lattice having a point at P, but there may be more, even infinitely many, and, as P moves along C, the corresponding critical lattices need not behave in any simple manner. I do not know if there is a general converse for non-convex K corresponding to that for convex K, but the following theorem sometimes applies.

THEOREM. *Let K be a star domain and $\delta > 0$ a constant. For each value of t in the range $0 \leqslant t \leqslant 1$, let there be a K-admissible lattice Λ_t of determinant δ with a basis P_t, Q_t such that the six points*

$$P_t, \quad Q_t, \quad R_t = Q_t-P_t, \quad \overline{P}_t, \quad \overline{Q}_t, \quad \overline{R}_t$$

lie on the boundary C of K in that order as C is described in a positive direction, and let the following conditions be satisfied:

(i) *as t increases continuously, the points P_t and Q_t (but not necessarily R_t) move continuously in a positive direction along C;*

(ii) $P_1 = Q_0$,

or, more generally,

(ii′) *there do not exist, on one and the same radius vector from O, two points L and M exterior to K such that L and M are inner points of the parallelograms $OP_0S_0Q_0$ and $OP_1S_1Q_1$ respectively, where $S_t = P_t+Q_t$;*

† M. Lemma 9. ‡ M. Lemma 4.

§ Mahler, *Proc. Kon. Nederland. Akad. Wet.* 49 (1946), 331–43, Theorem C.

and

(iii) *for each t the line-segments $Q_t R_t$ and $\bar{R}_t P_t$ lie in K.*

Then $\Delta(K) = \delta$. Further, if $\Lambda^{(c)}$ is any critical lattice, there is some value of t such that a basis for $\Lambda^{(c)}$ is of one of the three types:

(*a*) *P_t and Q_t, so $\Lambda^{(c)} = \Lambda_t$;*

(*b*) *P_t and an inner point, F say, of $Q_t R_t$ which is also a boundary point of K;*

(*c*) *Q_t and an inner point, G say, of $\bar{R}_t P_t$ which is a boundary point of K.*

Note that we do not state that the irreducibility of K follows from the conditions laid down; and, in fact, it is easy to construct a reducible K to which the theorem applies.† However, as the theorem gives us a complete inventory of critical lattices, we are in a position to use the general criteria of Mahler‡ in any particular case.

The theorem is sometimes available to confirm an intelligent guess at the critical lattices of a given star domain. I shall sketch shortly three examples, but refrain from discussing the question of reducibility further.

COROLLARY I. *Let $K^{(1)}$ be the unit circle*

$$x^2+y^2 \leqslant 1,$$

then $\Delta(K^{(1)}) = \frac{1}{2}\sqrt{3}$, and all critical lattices have as basis two points P and Q at an angular distance $\frac{1}{3}\pi$ apart on the circumference.§

In this case, we take $P_t Q_t R_t \bar{P}_t \bar{Q}_t \bar{R}_t$ to be the inscribed regular hexagon for which OP_t makes an angle $\frac{1}{3}t\pi$ with the positive x-axis. Conditions (i), (ii), (iii) are satisfied, and only critical lattices of the type (*a*), i.e. just the Λ_t, can arise. Since $K^{(1)}$ is convex, this corollary also follows from Mahler's work.

† If k is integral, the lattices Λ_t of Corollary II are not merely $K^{(2)}$-admissible but $K^{(4)}$-admissible, where $K^{(4)}$ is any (bounded) star domain included in the infinite region $-1 \leqslant xy \leqslant k$. If $K^{(4)}$ includes $K^{(2)}$, but is not identical with it, the lattices Λ_t still satisfy (i), (ii), (iii) and the general conditions of the theorem. As $\Delta(K^{(4)}) = \Delta(K^{(2)})$, the star domain $K^{(4)}$ is, by definition, reducible.

‡ Mahler, *Proc. London Math. Soc.* (2) 49 (1946), 137–58. There is a detailed discussion of the case $k = 1$ for the $K^{(2)}$ of Corollary II in Mahler, *Proc. Cambridge Phil. Soc.* 40 (1944), 107–16.

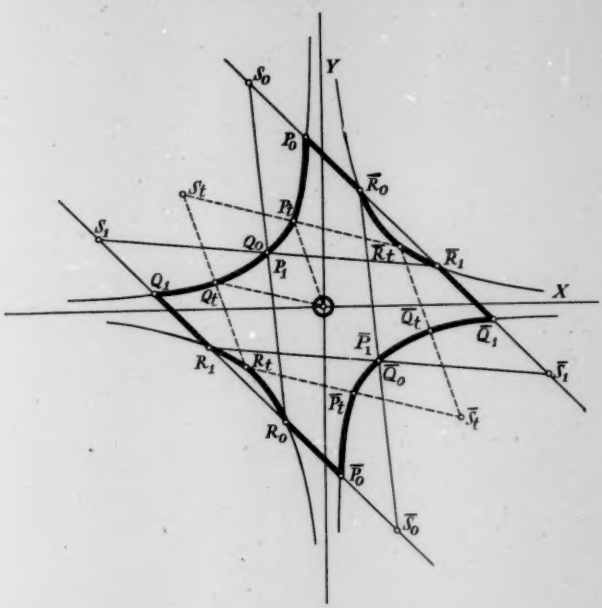
§ The result is, of course, classical.

COROLLARY II.† *Let k be a positive constant, and let $K^{(2)}$ be the star domain‡ defined by*

$$-1 \leqslant xy \leqslant k$$

and

$$|x+y| \leqslant (k^2+4k)^{\frac{1}{2}} = \delta \text{ (say)}.$$



Then

$$\Delta(K^{(2)}) = \delta = (k^2+4k)^{\frac{1}{2}},$$

and the only critical lattices are those generated by the point $(-1, 1)$ and any point on $x+y = \delta$, together with the lattices, Λ_t, of basis

$$\left.\begin{aligned} P_t &= (-g^{t-1}, g^{-t+1}) \\ Q_t &= (-g^t, g^{-t}) \end{aligned}\right\},$$

where

$$g = \tfrac{1}{2}(k+2+\delta), \qquad g^{-1} = \tfrac{1}{2}(k+2-\delta).$$

† The result, for general k, is due to N. Mullineux in a forthcoming paper. I am obliged to Mr. Mullineux for letting me see this paper before publication. The case $k = 1$ is due to Mahler, *Proc. Cambridge Phil. Soc.* 40 (1944), 107–16. If k is general, but no bound is put on $|x+y|$, the result is due to Segre, *Duke Math. J.* 12 (1945), 337–65, and, finally, if $k = 1$ and there is no bound on $|x+y|$, the result is classical.

‡ See diagram.

If we describe the boundary of $K^{(2)}$ positively, starting at P_0, the intersection of $x+y=\delta$ with $xy=-1$ in the second quadrant, we find the following boundary arcs:

A_1: the arc of $xy=-1$ from P_0 to Q_1, the image of P_0 in $x+y=0$; the mid-point of A_1 is $Q_0=P_1=(-1,1)$.

A_2: the segment of $x+y=-\delta$ between Q_1 and R_1, the intersection of the line with $xy=k$ which is nearer the x-axis.

A_3: the arc of $xy=k$ from R_1 to R_0, the image of R_1 in $x-y=0$.

A_4: the image of A_2 in $x-y=0$,

together with $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, and $\bar{A}_4$.

It may be verified that P_t and Q_t are the basis for a lattice Λ_t of the required kind, and that, as t increases continuously from 0 to 1, the point P_t moves continuously and positively along A_1 from P_0 to P_1, the point Q_t moves continuously and positively along A_1 from $P_1=Q_0$ to Q_1, and the point R_t moves continuously but negatively along A_3 from R_0 to R_1. Conditions (i) and (ii) are thus fulfilled, and it may be verified that (iii) is also true. Indeed, critical lattices $\Lambda^{(c)}$ of types (b) and (c) can arise only if $t=1$ or 0 respectively, so that, in both cases, $P_1=Q_0=(-1,1)$ is a point of $\Lambda^{(c)}$.

COROLLARY III. *Let k be any constant in the range*

$$0<k\leqslant\tfrac{1}{2},$$

and let $K^{(3)}$ be the star domain† defined by

$$\left.\begin{aligned}|x|&\leqslant 1\\ |y|&\leqslant 1+k|x|(1-2|x|)\end{aligned}\right\}.$$

Then $\Delta(K^{(3)})=1$, and the only critical lattices are those generated by $(0,1)$ and any point on $x=1$, together with the lattices Λ_t having as basis

$$\left.\begin{aligned}P_t&=\{1,\ k(2t-1)\}\\ Q_t&=\{(1-t),\ 1+k(2t-1)(1-t)\}\end{aligned}\right\}.$$

If we describe the boundary C of $K^{(3)}$ positively, starting with $\bar{R}_1=(1,-1+k)$, we have the following boundary arcs:

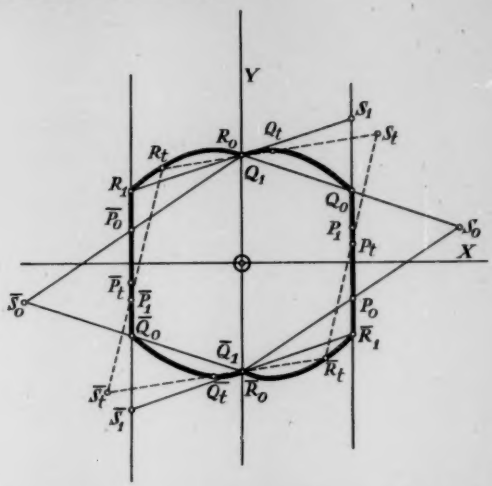
A_1: the line $x=1$ from $\bar{R}_1=(1,-1+k)$, to $Q_0=(1,1-k)$;

A_2: the arc of $y=1+kx(1-2x)$ between $Q_0=(1,1-k)$ and $R_0=Q_1=(0,1)$;

† See diagram.

A_3: the arc of $y = 1-kx(1+2x)$ between $R_0 = Q_1 = (0, 1)$ and $R_1 = (-1, 1-k)$,

together with $\bar{A}_1$, $\bar{A}_2$, and $\bar{A}_3$.



It is easily verified that P_t and Q_t are the basis of a $K^{(3)}$-admissible lattice Λ_t. As t increases from 0 to 1, P_t moves positively along a central portion of A_1 from $P_0 = (1, -k)$ to $P_1 = (1, +k)$, and Q_t, R_t move positively along A_2, A_3 respectively. Hence (i) is satisfied, and it is at once apparent from the diagram that (ii′) is satisfied. Condition (iii) also holds, and, indeed, critical lattices of type (*b*) can only arise if $0 < t < 1$ and $F = (0, 1)$, while those of type (*c*) require $t = 1$. Since $Q_1 = (0, 1)$, this point is thus always a point of the critical lattices distinct from the Λ_t.

2. The classical lemmas

We require two classical lemmas, which I quote without proof.

LEMMA 1. *Let Λ be H-admissible, where H is the parallelogram $\bar{R}SR\bar{S}$ of area $4d(\Lambda)$ and centre at the origin O. If P and Q are the mid-points of $\bar{R}S$ and SR respectively, there is a basis for Λ of one of the following types:*

(*a*) *P and Q;*

(*b*) *P and an inner point F of QR;*

(*c*) *Q and an inner point G of $\bar{R}P$.*

LEMMA 2. *Let $OPSQ$ be a parallelogram of area $d(\Lambda)$, and let the points M and N of Λ lie in $OPSQ$. Then, if at least one of the two points is an inner point of the parallelogram, they both lie on the same radius vector from O.*

3. Proof of the theorem

We have to show that every K-admissible lattice Λ of determinant $d(\Lambda) = \delta' \leqslant \delta$ is of one of the types (a), (b), (c) of the theorem. If we had $\delta' < \delta$, the lattice Λ' obtained by multiplying the coordinates of all the points of Λ by

$$\sqrt{(\delta/\delta')} > 1,$$

would be of determinant $d(\Lambda') = \delta$, and K-admissible since K is a star domain. Λ' could not be of one of the types (a), (b), (c) of the theorem, as these all have points on the boundary C of K. Hence we may confine ourselves to the case $d(\Lambda) = \delta$.

Let S_t be the point $S_t = P_t + Q_t$ so that $\bar{R}_t S_t R_t \bar{S}_t$ is a parallelogram, H_t say, of area $4d(\Lambda_t) = 4\delta = 4d(\Lambda)$, with the origin O as centre and the points P_t, Q_t as the mid-points of $\bar{R}_t S_t$ and $S_t R_t$ respectively. On applying a suffix t throughout to Lemma 1 we obtain

LEMMA 3. *Let Λ be a K-admissible lattice of determinant $d(\Lambda) = \delta$ which is H_t-admissible for some t. Then Λ is of one of the types (a), (b), (c) of the theorem.*

The types (a), (b), (c) of the theorem correspond to the types (a), (b), (c) respectively of Lemma 1. We need only observe that the points F and G of Lemma 1 are points of K by (iii) of the theorem, but cannot be inner points of K, since Λ is K-admissible.

The object of the next three lemmas is to show that every K-admissible Λ is H_t-admissible, for some t, if $d(\Lambda) = \delta$.

LEMMA 4. *Condition* (ii) *of the theorem implies* (ii′).

For, if $P_1 = Q_0$, the parallelograms $OP_0S_0Q_0$ and $OP_1S_1Q_1$ have the side $OP_1 = OQ_0$ in common but lie on opposite sides of it.

LEMMA 5. *If Λ is K-admissible but not H_t-admissible and $\Delta(\Lambda) = \delta$, then a point of Λ is an inner point of the parallelogram $OP_tS_tQ_t$.*

For, by hypothesis, there is a point N of Λ which is an inner point of H_t but does not belong to K. Since both K and H_t are star domains, and the only portions of the boundary of H_t which do not lie in K are the lines S_tP_t, S_tQ_t (but excluding P_t, Q_t) and their images

in the origin, the radius vector ON must lie between OP_t and OQ_t or their images in the origin; and we may assume that the former is the case.

LEMMA 6. *If Λ is K-admissible and of determinant $d(\Lambda) = \delta$, then it is H_t-admissible for some t.*

If Λ is H_0-admissible, there is nothing to prove. If not, by Lemma 5, there is a point, L say, of Λ which is an inner point of the parallelogram $OP_0S_0Q_0$. There may be several such points, but, if so, they all lie on the same radius vector ρ from O, by Lemma 2. Now, by (ii'), or (ii) and Lemma 4, no point of ρ, and *a fortiori* no point of Λ on ρ, is an inner point of $OP_1S_1Q_1$; and so, as t increases continuously from 0 to 1 and consequently as P_t and Q_t move continuously along the boundary of K, there will be a well-defined value, τ say, of t where for the last time a point of Λ on ρ is leaving the parallelogram $OP_tS_tQ_t$; that is, a point M (say) of Λ on ρ lies on the boundary of $OP_\tau S_\tau Q_\tau$ but no point of Λ on ρ is an inner point of $OP_tS_tQ_t$ for $t \geqslant \tau$. No point, N say, of Λ can be an inner point of $OP_\tau S_\tau Q_\tau$, for N would necessarily lie on ρ, by Lemma 2 applied to $OP_\tau S_\tau Q_\tau$. Hence, by Lemma 5, Λ is H_τ-admissible, which proves Lemma 6.

The theorem now follows immediately from Lemmas 3 and 6.

In conclusion, I should like to thank Prof. L. J. Mordell for his guidance in the writing of this paper.

ON AN INTEGRAL EQUATION OF CHANDRASEKHAR

By M. M. CRUM (*Oxford*)

[Received 28 May 1947]

1. Introduction. In certain problems of radiative equilibrium in a stellar atmosphere Chandrasekhar (**1**) has encountered integral equations of the form

$$\frac{1}{H(z)} = 1 - \int_0^1 \frac{z}{z+t} H(t)\, d\alpha(t), \tag{1}$$

where $\alpha(t)$ is a given function, and $H(t)$ the unknown. The equation (1) holds primarily for $0 < z < 1$ but it defines $H(z)$ also for complex z. Chandrasekhar (**2**) has obtained various properties of the solution $H(z)$, including

$$\frac{1}{H(z)H(-z)} = A(z), \tag{2}$$

where $A(z)$ is the known function defined except for $-1 \leqslant z \leqslant 1$ by

$$A(z) = 1 - \int_0^1 \frac{2z^2}{z^2-t^2}\, d\alpha(t).$$

Chandrasekhar has conjectured that a solution of (1) is given by

$$\log H(z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \log A(w) \frac{z\, dw}{w^2-z^2}. \tag{3}$$

If it is assumed that a solution of (1) exists which is analytic and non-zero for $R(z) \geqslant 0$, then (3) will follow from (2) by Cauchy's theorem.

Together (1) and (2) give the linear equation

$$H(z)\left\{1 - \int_0^1 \frac{z\, d\alpha(t)}{z+t}\right\} = 1 + \int_0^1 \frac{z}{z-t} \{H(z)-H(t)\}\, d\alpha(t). \tag{4}$$

If we assume that

$$H(z) = \int_0^\infty h(u)e^{-uz}\,du,$$

we can obtain from (4) an integral equation for $h(u)$, of the form

$$h(u) = \int_0^\infty h(v)g(u-v)\,dv. \tag{5}$$

An equation of type (5) was obtained by Milne directly from astrophysical considerations, and the general method of solution was developed by Hopf and Wiener (3), (4); the solution is ultimately equivalent to (3). It is, however, of interest to attack (1), (2), and (4) directly, and their relation to (5) is not discussed here. In this note it is proved that, if

(i) $$\int_0^1 |d\alpha(t)| < \infty, \qquad \int_0^\delta |d\alpha(t)| \to 0 \quad \text{as} \quad \delta \to 0,$$

then the general solution of (1) such that

(ii) $$|H(t)| < M \quad (0 \leqslant t \leqslant 1)$$

is given (for values of z specified below) by

$$\log H(z) = \frac{1}{2\pi i}\int_C \log A(w)\frac{2z}{w^2-z^2}\,dw. \tag{6}$$

Here C may be any contour which goes straight from 0 to $i\delta$ ($\delta > 0$), and thence to infinity avoiding all singularities of the integrand (other than ∞). C and its reflection in the origin divide the z-plane into two or more regions, and z must lie in the same region as all small-enough positive numbers; but $H(z)$ has a single-valued continuation outside this region. If $A(w)$ has no zeros, then $H(z)$ is unique; otherwise different solutions can be obtained by changing C, and any two such solutions are related by an equation

$$H_2(z) = \prod_{\nu=1}^{N}\left(\frac{z_\nu+z}{z_\nu-z}\right)H_1(z),$$

where the z_ν are zeros of $A(z)$.

2. Let $H(z)$ be a solution of (1) and let the conditions (i) and (ii) be satisfied. Then, by (1), $1/H(z)$ is analytic at infinity and in the

z-plane slit from -1 to 0. Except for $-1 < z < 1$, by absolute convergence, using (1) repeatedly, we have

$$\left\{1-\frac{1}{H(z)}\right\}\left\{1-\frac{1}{H(-z)}\right\}$$

$$= \int_0^1\int_0^1 \frac{z}{z+s}\frac{z}{z-t}H(s)H(t)\,d\alpha(s)\,d\alpha(t)$$

$$= \int_0^1\int_0^1 \left(\frac{t}{t+s}\frac{z}{z-t}+\frac{s}{s+t}\frac{z}{z+s}\right)H(s)H(t)\,d\alpha(s)\,d\alpha(t)$$

$$= \int_0^1 \frac{z}{z-t}H(t)\left\{1-\frac{1}{H(t)}\right\}d\alpha(t)+\int_0^1 \frac{z}{z+s}H(s)\left\{1-\frac{1}{H(s)}\right\}d\alpha(s)$$

$$= 1-\frac{1}{H(-z)}+1-\frac{1}{H(z)}-\int_0^1 \frac{z\,d\alpha(t)}{z-t}-\int_0^1 \frac{z\,d\alpha(s)}{z+s},$$

which reduces to (2).

Still excepting the case $-1 < z < 1$, substituting for $1/H(z)$ from (2), we can rewrite (1) in the form

$$H(-z)\left\{1-\int_0^1 \frac{z}{z-t}\,d\alpha(t)-\int_0^1 \frac{z}{z+t}\,d\alpha(t)\right\} = 1-\int_0^1 \frac{z}{z+t}H(t)\,d\alpha(t),$$

which is (4) with $-z$ instead of z. Since both sides of (4) are analytic for $R(z) > 0$, (4) holds also for $0 < z \leqslant 1$.

Conversely, (2) and (4) together imply (1).

3. We thus require to deduce (2) and (4) from (6), and (6) from (2) and (4).

The function $A(z)$ is analytic at infinity and in the z-plane slit from -1 to 1. As $y \to 0$, $A(iy) \to 1$ since

$$1-A(iy) = O\left\{\int_0^\delta |d\alpha(t)|\right\}+O\left\{\frac{y^2}{\delta^2}\int_\delta^1 |d\alpha(t)|\right\},$$

which can be made arbitrarily small by choosing first δ and then y. As $y \to \infty$, either $A(iy)$ tends to a non-zero limit, or $(iy)^{2n}A(iy)$ does

so for some positive integer n. Hence $A(z)$ has, on the imaginary axis, at most a finite number of zeros, say $\pm iy_\nu$, where $y_\nu > 0$.

Let
$$B(z) = A(z)(1-z^2)^n \prod_\nu \left(\frac{1-z^2}{1+z^2/y_\nu^2}\right). \tag{7}$$

Then $B(z)$ is analytic and non-zero on the imaginary axis (and at infinity) except perhaps at $z = 0$; since $B(iy) \to 1$ as $y \to 0$, $\log B(iy)$ is bounded.

For $R(z) > 0$ we define
$$\log K(z) = \frac{1}{2\pi i}\int_0^{i\infty} \log B(w)\frac{2z\,dw}{w^2-z^2}, \tag{8}$$

where $\log B(iv) \to 0$ as $v \to 0$. Then $\log K(z)$ is analytic for $R(z) > 0$; we can obtain a unique analytic continuation of $K(z)$ across the imaginary axis by deforming the contour; e.g. when z is in the second quadrant and $B(z) \neq 0$,

$$\begin{aligned}\log K(z) &= \frac{1}{2\pi i}\int_{(z-)} \log B(w)\frac{2z\,dw}{w^2-z^2}\\ &= -\log B(z) - 2k\pi i + \frac{1}{2\pi i}\int_0^{i\infty} \log B(w)\frac{2z\,dw}{w^2-z^2}\\ &= -\log B(z) - 2k\pi i - \log K(-z)\end{aligned}$$

since $R(-z) > 0$. If z passes from the fourth to the third quadrant the contour must be deformed so as to pass to the right of $-z$, and the same result is obtained. Thus $K(z)$ is single-valued, and, for all z,

$$\frac{1}{K(z)K(-z)} = B(z).$$

Now let
$$H_1(z) = K(z)(1+z)^n \prod_\nu \frac{1+z}{1-z/iy_\nu}, \tag{9}$$

so that
$$\frac{1}{H_1(z)H_1(-z)} = A(z). \tag{10}$$

I shall prove that $H_1(z)$ satisfies (6), (1), and (4). Let C_1 consist of the imaginary axis from 0 to $i\infty$, indented so as to pass to the right of each point iy_ν; and let C_1' be the reflection of C_1 in the origin,

described from $-i\infty$ to 0. Then, if z and $-\zeta$ are to the right of $C_1'+C_1$,

$$\frac{1}{2\pi i}\int_{C_1}\log\left(1-\frac{w^2}{\zeta^2}\right)\frac{2z\,dw}{w^2-z^2}=\frac{1}{2\pi i}\int_{C_1'+C_1}\log\left(1-\frac{w}{\zeta}\right)\frac{2z\,dw}{w^2-z^2}$$
$$=-\log\left(1-\frac{z}{\zeta}\right)$$

by removing the contour to the right. Hence

$$\log H_1(z)$$
$$=\frac{1}{2\pi i}\int_{C_1}\left\{\log A(w)+n\log(1-w^2)+\sum_\nu\log\left(\frac{1-w^2}{1+w^2/y_\nu^2}\right)\right\}\frac{2z\,dw}{w^2-z^2}+$$
$$+n\log(1+z)+\sum_\nu\log\left(\frac{1+z}{1-z/iy_\nu}\right)$$
$$=\frac{1}{2\pi i}\int_{C_1}\log A(w)\frac{2z\,dw}{w^2-z^2}. \tag{11}$$

4. Now, if $z=x+iy=re^{i\theta}$, then from (8), for $x>0$,

$$\log K(z)=O\left\{\int_0^\infty\frac{r\,dv}{|r^2e^{2i\theta}+v^2|}\right\}$$
$$=O\left\{\int_0^\infty\frac{dv_1}{|e^{2i\theta}+v_1^2|}\right\}\quad(v=rv_1)$$
$$=O(1)+O\left\{\int_{-2}^{2}\frac{dv_1}{|v_1+ie^{i\theta}|}\right\}$$
$$=O(1)+O\left\{\int_{-2}^{2}\frac{dv_1}{|v_1-\sin\theta|+\cos\theta}\right\}$$
$$=O(1)+O\left\{\int_0^2\frac{dv_2}{v_2+\cos\theta}\right\}\quad(v_2=v_1-\sin\theta)$$
$$=O(1)+O(\log\sec\theta).$$

That is, for some constants κ and λ,

$$|\log|K(z)||\leqslant|\log K(z)|\leqslant\kappa+\lambda\log\sec\theta,$$
$$K(z)=O(\sec\theta)^\lambda,\qquad 1/K(z)=O(\sec\theta)^\lambda,$$

and so, for large enough μ,

$$H_1(z) = O\{(1+r^n)\sec^\mu\theta\}, \qquad 1/H_1(z) = O(\sec^\mu\theta).$$

Also, when $r \to 0$ with $|\theta| \leqslant \frac{1}{4}\pi$, $K(z) \to 1$. For

$$\begin{aligned}\log K(z) &= O\left\{\int_0^\infty |\log B(iv)|\frac{r\,dv}{r^2+v^2}\right\}\\ &= O\left\{\int_0^\delta |\log B(iv)|\frac{r\,dv}{r^2+v^2}\right\}+O\left\{\int_\delta^\infty \frac{r\,dv}{r^2+v^2}\right\}\\ &= o\left\{\int_0^\delta \frac{r\,dv}{r^2+v^2}\right\}+O\left\{r\int_\delta^\infty \frac{dv}{v^2}\right\}\\ &= o(1)+O\left(\frac{r}{\delta}\right) = o(1)\end{aligned}$$

by choosing first δ and then r small enough.

Hence also $H_1(z) \to 1$ as $r \to 0$, $|\theta| \leqslant \frac{1}{4}\pi$.

5. Now let

$$\phi(z) = H_1(z)\left\{1-\int_0^1 \frac{z}{z+t}\,d\alpha(t)\right\}-1-\int_0^1 \frac{z}{z-t}\{H_1(z)-H_1(t)\}\,d\alpha(t), \tag{12}$$

so that, by (10),

$$\phi(-z) = \frac{1}{H_1(z)}-1+\int_0^1 \frac{z}{z+t}H_1(t)\,d\alpha(t). \tag{13}$$

From (13), $\phi(z)$ is analytic for $x \leqslant 0$ except perhaps at $z = 0$, and for $x < 0$, by § 4,

$$\phi(z) = O\{|\sec\theta|^\mu\}+O(1) = O\{|\sec\theta|^\mu\}.$$

From (12), $\phi(z)$ is analytic for $x > 0$, and, for $y \neq 0$, by § 4,

$$\begin{aligned}\phi(z) &= O\left\{(1+r^n)\sec^\mu\theta\left(1+\int_0^1 \left|\frac{r}{y}\frac{d\alpha(t)}{y}\right|\right)\right\}\\ &= O\{(1+r^n)\sec^\mu\theta\,\mathrm{cosec}\,\theta\}\\ &= O\{(1+r^n)\mathrm{cosec}^\mu 2\theta\}\end{aligned}$$

if we choose $\mu \geqslant 1$.

For $r = \frac{1}{2}R$, and for $r = 2R$,

$$\left|\left(1-2^4\frac{z^4}{R^4}\right)^{\mu}\left(1-\frac{1}{2^4}\frac{z^4}{R^4}\right)^{\mu}\phi(z)\right| < C(1+R^n),$$

where C is a constant, independent of R. By the maximum-modulus theorem the inequality holds also for $r = R$, so that

$$\phi(z) = O(1+r^n).$$

Hence, in fact, $\phi(z)$ is analytic at $z = 0$, and is a polynomial. But from (13), $\phi(-z)$ is bounded for $\theta = \frac{1}{4}\pi$, and tends to zero as $r \to 0$. Hence $\phi(z) \equiv 0$, and (12) and (13) state that $H_1(z)$ is a solution of (4) and (1).

6. Now let $H(z)$ be any solution of (1) which is bounded for $0 \leqslant z \leqslant 1$. We have to verify that $H(z)$ is given by (6). By (1), $1/H(z)$ is analytic except for $-1 \leqslant z \leqslant 0$, and bounded for $x \geqslant 0$. Also (2) holds, so that, if

$$\psi(z) = H(z)/H_1(z),$$

then, by (2) and (10),

$$\psi(z)\psi(-z) = 1. \tag{14}$$

Since $\psi(z)$ is meromorphic for $x \geqslant 0$ except perhaps at $z = 0$, $\psi(z)$ is meromorphic everywhere except perhaps at $z = 0$; but $\psi(z) \to 1$ as $z \to 0$ with $x \geqslant 0$, by (1), hence also as $z \to 0$ with $x < 0$; hence $\psi(z)$ is analytic at $z = 0$. But it is analytic at infinity because, by (2) and (10), $H(z)$ and $H_1(z)$ have poles of the same order. Hence $\psi(z)$ is rational and

$$\psi(z) = \frac{\prod(1-z/z'_\nu)}{\prod(1-z/z_\nu)}.$$

By (14),

$$\prod\left(1-\frac{z^2}{z_\nu^2}\right) \equiv \prod\left(1-\frac{z^2}{z_\nu'^2}\right),$$

and we can suppose $z'_\nu = -z_\nu$. Thus

$$H(z) = H_1(z)\prod\prod\left(\frac{z_\nu+z}{z_\nu-z}\right). \tag{15}$$

Since both $H(z)$ and $H_1(z)$ are bounded and non-zero for $0 \leqslant z \leqslant 1$, none of the z_ν can lie on the line $-1 \leqslant z \leqslant 1$; also the z_ν are poles of $H(z)$ and of $H_1(-z)$, and zeros of $A(z)$ by (2).

Conversely, let $H_n(z)$ be any solution of (1), and let

$$H_{n+1}(z) = \frac{z_n+z}{z_n-z}H_n(z),$$

where $-z_n$ is a pole of $H_n(z)$. Then

$$\int_0^1 \frac{z}{z+t} H_{n+1}(t)\, d\alpha(t)$$

$$= \int_0^1 \frac{z(z_n+t)}{(z+t)(z_n-t)} H_n(t)\, d\alpha(t)$$

$$= \frac{z_n-z}{z_n+z} \int_0^1 \frac{z}{z+t} H_n(t)\, d\alpha(t) + \frac{2z}{z_n+z} \int_0^1 \frac{z_n}{z_n-t} H_n(t)\, d\alpha(t)$$

$$= \frac{z_n-z}{z_n+z}\left(1-\frac{1}{H_n(z)}\right)+\frac{2z}{z_n+z} = 1-\frac{1}{H_{n+1}(z)},$$

so that $H_{n+1}(z)$ is a solution of (1). Hence, by induction, (15) gives a solution of (1) if the z_ν are any poles of $H_1(-z)$.

On the other hand, if $H(z)$ and $H_1(z)$ are defined by (6), with contours C and C_1, then for z small and positive

$$\log H(z) - \log H_1(z) = \frac{1}{2\pi i} \int_{C-C_1} \log A(w)\left(\frac{1}{w-z} - \frac{1}{w+z}\right) dw$$

$$= \frac{1}{2\pi i} \int_{C-C_1} \frac{A'(w)}{A(w)} \log\frac{w+z}{w-z}\, dw$$

(where that branch of the logarithm is taken which is $O(1/w)$ at infinity)

$$= \sum_\nu \log\frac{z_\nu+z}{z_\nu-z}$$

by Cauchy's theorem, where the z_ν are zeros of $A(z)$. The use of Cauchy's theorem is justified if z is small and positive, for we may then withdraw $C-C_1$ to infinity without encountering the points $w = \pm z$. Thus (15) follows by analytic continuation. Conversely, (6) follows from (15) if C is constructed to surround with C_1 the required zeros z_ν.

7. It is presumably false that $A(z)$ can have a finite number of zeros only, so that we might expect an infinite product in (15). Such a product could converge if the factors were paired and if $z_\nu \to 0$. But $H(z)$ would then have an infinity of poles near the line $0 < z < 1$ and $H(t)$ would not be bounded for small positive t.

In the astrophysical equations of Chandrasekhar, $\alpha(t)$ is real and increasing, and either

$$\beta = \int_0^1 d\alpha(t) = \tfrac{1}{2}, \quad \text{or} \quad \beta = \int_0^1 d\alpha(t) < \tfrac{1}{2}.$$

If $z^2 = u+iv$,

$$A(z) = 1-2\beta-2\int_0^1 \frac{(u-t^2)-iv}{(u-t^2)^2+v^2}\, t^2\, d\alpha(t).$$

The imaginary part vanishes only for $v = 0$, and then

$$A(z) = 1-2\beta-2\int_0^1 \frac{t^2\, d\alpha(t)}{u-t^2},$$

which is monotonic increasing for $u < 0$ or $u > 1$. If $\beta = \frac{1}{2}$ the only zero of $A(z)$ is at infinity, and $H(z)$ is unique and is given by (3). If $\beta < \frac{1}{2}$, $A(\infty) > 0$, and there may be a pair of real zeros $\pm x_0$, where $x_0 = \sqrt{u_0} > 1$. There are now two solutions of (1), $H_1(z)$ as given by (3), and

$$H_2(z) = \frac{x_0+z}{x_0-z} H_1(z).$$

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ON nth-POWER RESIDUES

By H. GUPTA (*Hoshiarpur*)

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1. THROUGHOUT this paper, p denotes a prime $\geqslant 2$; all small letters except x and y denote integers $\geqslant 0$, unless stated otherwise; and ϕ is Euler's function.

I obtain, in this paper, some identical congruences* connected with the numbers $s(m)$, the nth-power residues of m which are less than and prime to m.

I make use of the result (3)

$$(x-1)(x-a)(x-a^2)\dots(x-a^{u-1}) = \sum_{r=0}^{u}(-1)^r\begin{bmatrix}u\\r\end{bmatrix}_a x^{u-r} \quad (a>1;\, u\geqslant 1), \tag{1.1}$$

where

$$\begin{bmatrix}u\\r\end{bmatrix}_a = \frac{(a^u-1)(a^u-a)(a^u-a^2)(a^u-a^3)\dots(a^u-a^{r-1})}{(a-1)(a^2-1)(a^3-1)(a^4-1)\dots(a^r-1)} \quad (r>0), \tag{1.2}$$

$$\begin{bmatrix}u\\0\end{bmatrix}_a = 1. \tag{1.3}$$

2. THEOREM 1. *If s runs through the n-th-power residues of p^λ ($\lambda>0; p\geqslant 3$), then for all x*

$$\prod_{p^\lambda}(x-s) \equiv (x^t-1)^{p^{\lambda-\alpha}} \pmod{p^\lambda},$$

where $t=(p-1)/(n,p-1)$, *and* $p^{\alpha-1}=(n,p^{\lambda-1})$ $(\alpha>0)$.

In particular $\prod_{p^\lambda} s \equiv (-1)^{t-1} \pmod{p^\lambda}$.

Let g be a primitive root of p^λ. Then the nth-power residues of p^λ are congruent modulo p^λ to the numbers of the set

$$1,\ g^\kappa,\ g^{2\kappa},\ \dots,\ g^{\phi(p^\lambda)-\kappa},$$

where $\kappa=\{n,\phi(p^\lambda)\}=p^{\alpha-1}h$, $h=(n,p-1)$.

Hence, for all x,

$$\prod_{p^\lambda}(x-s) \equiv \prod_{j=0}^{(p^{\lambda-\alpha}t-1)}(x-g^{\kappa j}) \pmod{p^\lambda}$$

$$\equiv \sum_{r=0}^{p^{\lambda-\alpha}t}(-1)^r\begin{bmatrix}p^{\lambda-\alpha}t\\r\end{bmatrix}_{g^\kappa} x^{p^{\lambda-\alpha}t-r} \pmod{p^\lambda}.$$

* Cf. Gupta (1), (2).

Now, we notice that

$$g^{\phi(p^\lambda)}-g^{\kappa v} \equiv 1-g^{\kappa v} \pmod{p^\lambda} \quad (v \geqslant 1).$$

Therefore, when $t \nmid v$

$$\frac{g^{\phi(p^\lambda)}-g^{\kappa v}}{g^{\kappa v}-1} \equiv -1 \pmod{p^\lambda} \tag{2.1}$$

and

$$\frac{g^{\phi(p^\lambda)}-1}{g^{\kappa v}-1} \equiv 0 \pmod{p^\lambda}. \tag{2.2}$$

When $t \mid v$, let $v = tp^{\beta-\alpha}v_1$, where $\beta \geqslant \alpha$ and $(v_1, p) = 1$. Then

$$g^{\kappa v} = g^{v_1 \phi(p^\beta)} = (1+p^\beta i)^{v_1}, \quad \text{where} \quad (i,p) = 1$$

$$\equiv 1+v_1 ip^\beta \pmod{p^{2\beta}};$$

while

$$g^{\phi(p^\lambda)} = (1+p^\beta i)^{p^{\lambda-\beta}}$$

$$\equiv 1+ip^\lambda \pmod{p^{\lambda+\beta}}.$$

Therefore $v_1\{g^{\phi(p^\lambda)}-1\} \equiv p^{\lambda-\beta}\{g^{\kappa v}-1\} \pmod{p^{\lambda+\beta}}$,

i.e., when $t \mid v$,

$$\frac{g^{\phi(p^\lambda)}-1}{g^{\kappa v}-1} \equiv \frac{p^{\lambda-\beta}}{v_1} \equiv \frac{p^{\lambda-\alpha}}{v/t} \pmod{p^\lambda}, \tag{2.3}$$

and

$$\frac{g^{\phi(p^\lambda)}-g^{\kappa v}}{g^{\kappa v}-1} \equiv \frac{p^{\lambda-\alpha}-v/t}{v/t} \pmod{p^\lambda}. \tag{2.4}$$

Using the results (2.1)–(2.4), we get

$$\left[\begin{matrix} p^{\lambda-\alpha}t \\ r \end{matrix}\right]_{g^\kappa} \equiv \begin{cases} 0 \pmod{p^\lambda} & \text{when} \quad t \nmid r, \\ (-1)^{r-r_1}\binom{p^{\lambda-\alpha}}{r_1} \pmod{p^\lambda} & \text{when } t \mid r \text{ and } r_1 = r/t. \end{cases} \tag{2.5}$$

The theorem readily follows.

3. THEOREM 2. *If s runs through the n-th-power residues of 2^λ and n is odd, then, for all x,*

$$\prod_{2^\lambda}(x-s) \equiv \begin{cases} (x^2-1)^{2^{\lambda-2}} \pmod{2^\lambda} & \text{when} \quad \lambda \geqslant 2, \\ (x-1) \pmod{2^\lambda} & \text{when} \quad \lambda = 1. \end{cases}$$

The second part of the theorem is obvious. As regards the first part, we notice that the nth-power residues of 2^λ, when n is odd and $\lambda \geqslant 2$, are congruent modulo 2^λ to the numbers of the set

$$\pm(1, 3, 3^2, 3^3, \ldots, 3^{(2^{\lambda-2}-1)}).$$

Hence, for all x,

$$\prod_{2^\lambda}(x-s) \equiv \prod_{j=0}^{(2^{\lambda-2}-1)}(x^2-3^{2j}) \pmod{2^\lambda}$$

$$\equiv \sum_{r=0}^{2^{\lambda-2}}(-1)^r \begin{bmatrix} 2^{\lambda-2} \\ r \end{bmatrix}_{3^2} x^{(2^{\lambda-1}-2r)} \pmod{2^\lambda}.$$

In this case, we notice that, if

$$v = 2^{\alpha-2}v_1, \quad \text{where} \quad (v_1, 2) = 1 \quad \text{and} \quad \alpha \geqslant 2,$$

then
$$3^{2v} = (1+2^{\alpha+1}i)^{v_1}, \quad \text{where} \quad (i, 2) = 1,$$
$$\equiv 1+iv_1 2^{\alpha+1} \pmod{2^{2(\alpha+1)}};$$

while
$$3^{2^{\lambda-1}} = (1+2^{\alpha+1}i)^{2^{\lambda-\alpha}}$$
$$\equiv 1+i2^{\lambda+1} \pmod{2^{\lambda+\alpha+1}}.$$

Hence
$$\frac{3^{2^{\lambda-1}}-1}{3^{2v}-1} \equiv \frac{2^{\lambda-\alpha}}{v_1} \equiv \frac{2^{\lambda-2}}{v} \pmod{2^\lambda} \tag{3.1}$$

and
$$\frac{3^{2^{\lambda-1}}-3^{2v}}{3^{2v}-1} \equiv \frac{2^{\lambda-2}-v}{v} \pmod{2^\lambda}.$$

Thus
$$\begin{bmatrix} 2^{\lambda-2} \\ r \end{bmatrix}_{3^2} \equiv \binom{2^{\lambda-2}}{r} \pmod{2^\lambda},$$

and the theorem is proved.

4. THEOREM 3. *If s runs through the n-th-power residues of 2^λ and n is even, then, for all x,*

$$\prod_{2^\lambda}(x-s) \equiv \begin{cases} (x-2^\alpha-1)^{2^{\lambda-\alpha-1}} \pmod{2^\lambda} & (\lambda > \alpha+1 \geqslant 3), \\ (x-1) \pmod{2^\lambda} & (\lambda = 1, 2, 3, \ldots, \alpha, \alpha+1), \end{cases}$$

where
$$2^{\alpha-1} = (n, 2^{\lambda-1}).$$

The first part of Theorem 3 can also be written

$$\prod_{2^\lambda}(x-s) \equiv (x-1)^{2^{\lambda-\alpha-1}}+2^{\lambda-1}(x-1)^{2^{\lambda-\alpha-1}-1} \pmod{2^\lambda}$$

(identically) when $\lambda > \alpha+1 \geqslant 3$.

As a particular case of Theorems 2 and 3, we have

$$\prod_{2^\lambda} s \equiv 1 \pmod{2^{\lambda-1}} \quad (\lambda > 0).$$

The second part of the theorem needs no proof.

When $\lambda > \alpha+1 \geqslant 3$, the nth-power residues of 2^λ are

$$1, \quad 2^{\alpha+1}+1, \quad 2.2^{\alpha+1}+1, \quad 3.2^{\alpha+1}+1, \quad \ldots, \quad 2^\lambda-2^{\alpha+1}+1.$$

The theorem is evidently true when $\lambda = \alpha+2$.

Suppose it true when $\lambda \leqslant \rho$, so that

$$\prod_{j=0}^{(2^{\lambda-\alpha-1}-1)} (x-2^{\alpha+1}j-1) = (x-2^{\alpha}-1)^{2^{\lambda-\alpha-1}}+2^{\lambda}f(x) \quad (\alpha+2 \leqslant \lambda \leqslant \rho),$$

$f(x)$ being an integral polynomial in x. Then

$$\prod_{j=0}^{(2^{\rho-\alpha}-1)} (x-2^{\alpha+1}j-1)$$

$$= \prod_{j=0}^{(2^{\rho-\alpha-1}-1)} (x-2^{\alpha+1}j-1) \prod_{j=0}^{(2^{\rho-\alpha-1}-1)} (y-2^{\alpha+1}j-1), \text{ where } y = x-2^{\rho},$$

$$= \{(x-2^{\alpha}-1)^{2^{\rho-\alpha-1}}+2^{\rho}f(x)\}\{(y-2^{\alpha}-1)^{2^{\rho-\alpha-1}}+2^{\rho}f(y)\}.$$

Since, identically, $2^{\rho}f(x) \equiv 2^{\rho}f(x-2^{\rho}) \pmod{2^{\rho+1}}$, we have, for all x,

$$\prod_{j=0}^{(2^{\rho-\alpha}-1)} (x-2^{\alpha+1}j-1) \equiv \{(x-2^{\alpha}-1)^{2^{\rho-\alpha-1}}+2^{\rho}f(x)\}^2 \pmod{2^{\rho+1}}$$

$$\equiv (x-2^{\alpha}-1)^{2^{\rho-\alpha}} \pmod{2^{\rho+1}};$$

and the theorem follows by induction.

Similar proofs could be given for Theorems 1 and 2.

5. If $m = p^{\lambda}m_1$, where $(m_1, p) = 1$ and $p \geqslant 2$, and $N(m)$ denotes the number of nth-power residues of m, then it is easy to show that

$$N(m) = N(p^{\lambda})N(m_1). \tag{5.1}$$

We are now able to state

THEOREM 4. *If s_1 and s_2 run through the n-th-power residues of m and p^{λ} respectively, then, identically in x,*

$$\prod_{m} (x-s_1) \equiv \Big\{ \prod_{p^{\lambda}} (x-s_2) \Big\}^{N(m_1)} \pmod{p^{\lambda}}.$$

If the nth-power residues of m be divided by p^{λ}, then we obtain as remainders the nth-power residues of p^{λ} each repeated $N(m_1)$ times. Hence the theorem.

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